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Research Article

On Sequences of Diophantine 3-Tuples generated through Euler and Bernoulli Polynomials

M. A. Gopalan[1,*], Sharadha Kumar[2]

[1]*Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India.*
[2]*Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India.*

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Abstract: This paper deals with the study of constructing sequences of diophantine triples (a,b,c) such that the product of any two elements of the set added by a polynomial with integer coefficient is a perfect square. Diophantine 3-tuples, Euler Polynomials , Bernoulli Polynomials.

***Keywords**:* Diophantine 3-tuples; Euler polynomials; Bernoulli polynomials.

1. Introduction

The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets have been studied by Diophantus. A set of m distinct positive integers $\{a_1, a_2, a_3, \ldots\ldots, a_m\}$ is said to have the property $D\ n\ , n \in \mathbb{Z} -\ 0$ if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$ or $1 \le j < i \le m$ and such a set is called a Diophantine m-tuple with property $D(n)$.

Many mathematicians have considered the construction of different formulations of diophantine triples with the property $D(n)$ for any arbitrary integer n [1] and also, for any linear polynomials in n. In this context, one may refer [2-12] for an extensive review of various problems on diophantine triples.

This paper aims at constructing sequences of diophantine triples where the product of any two members of the triple with the polynomial with integer coefficients satisfies the required property.

2. The Definitions of Bernoulli and Euler polynomials

The Bernoulli and Euler polynomials can explicitly be defined as,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k$$

where B_k stands for Bernoulli numbers and

$$E_n(x) = \sum_{k=1}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$$

*Correspondence: mayilgopalan@gmail.com

where E_k stands for Euler numbers.

For the sake of clear understanding a few Bernoulli and Euler Polynomials are presented below:

Bernoulli Polynomials	**Euler Polynomials**
$B_0(x)=1$	$E_0(x)=1$
$B_1(x)=x-\frac{1}{2}$	$E_1(x)=x-\frac{1}{2}$
$B_2(x)=x^2-x+\frac{1}{6}$	$E_2(x)=x^2-x$
$B_3(x)=x^3-\frac{3}{2}x^2+\frac{x}{2}$	$E_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{4}$
$B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$	$E_4(x)=x^4-2x^3+x$

3. Method of Analysis:

Consider the Euler polynomial $E_2(x)$ and Bernoulli polynomial $B_2(x)$ given by

$$E_2(x)=x^2-x\ ,\quad B_2(x)=x^2-x+\frac{1}{6}$$

Let

$$a=E_2(x)=x^2-x,\quad b=6B_2(x)=6x^2-6x+1$$

It is observed that

$$ab+3x^4+3x^2-3x+1=(3x^2-2x+1)^2$$

Therefore, the pair (a,b) represents diophantine 2-tuple with the property $D(3x^4+3x^2-3x+1)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1+3x^4+3x^2-3x+1=p^2 \quad (1)$$

$$bc_1+3x^4+3x^2-3x+1=q^2 \quad (2)$$

Eliminating c_1 between (1) and (2), we have

$$bp^2-aq^2=(b-a)(3x^4+3x^2-3x+1) \quad (3)$$

Introducing the linear transformations

$$p=X+aT\ ,\ q=X+bT \quad (4)$$

in (3) and simplifying we get

$$X^2=abT^2+3x^4+3x^2-3x+1$$

which is satisfied by $T=1$, $X=3x^2-2x+1$.

In view of (4) and (1), it is seen that

$$c_1 = 13x^2 - 11x + 3$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2) where

$$c_2 = 22x^2 - 18x + 5$$

exhibits diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3) where

$$c_3 = 33x^2 - 27x + 7$$

exhibits diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4) where

$$c_4 = 46x^2 - 38x + 9$$

exhibits diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by $(a, c_s, c_{s+1})$where

$$c_s = \; s^2+6s+6 \; x^2 - \; s^2+4s+6 \; x + 2s+1 , \quad (s=1,2,3,...)$$

A few numerical examples are presented in Table 1 below:

Table 1: Numerical Examples

x	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)	(a, c_4, c_5)
2	55	(2,33,57)	(2,57,85)	(2,85,117)	(2,117,153)
3	262	(6,87,149)	(6,149,223)	(6,223,309)	(6,309,407)
4	805	(12,167,285)	(12,285,427)	(12,427,593)	(12,593,783)
5	1936	(20,273,465)	(20,465,697)	(20,697,969)	(20,969,1281)

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple (b, c_1, c_2) where

$$c_2 = 37x^2 - 33x + 8$$

exhibits diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3) where

$$c_3 = 73x^2 - 67x + 15$$

exhibits diophantine 3-tuple with property $D(3x^4+3x^2-3x+1)$.

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4) where

$$c_4 = 121x^2 - 113x + 24.$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (b, c_s, c_{s+1}) where

$$c_s = \ 6s^2 + 6s + 1 \ x^2 - \ 6s^2 + 4s + 1 \ x + s^2 + 2s \,, \quad (s = 1, 2, 3, \ldots).$$

A few numerical examples are presented in Table: 2 below:

Table: 2 Numerical Examples

x	D	(b, c_1, c_2)	(b, c_2, c_3)	(b, c_3, c_4)	(b, c_4, c_5)
2	55	(13,33,90)	(13,90,173)	(13,173,282)	(13,282,417)
3	262	(37,87,242)	(37,242,471)	(37,471,774)	(37,774,1151)
4	805	(73,167,468)	(73,468,915)	(73,915,1508)	(73,1508,2247)
5	1936	(121,273,768)	(121,768,1505)	(121,1505,2484)	(121,2484,3705)

For simplicity and brevity, some more sequences of 3-tuples generated through Euler and Bernoulli polynomials are presented in Table 3 below:

Table 3: Sequences of 3-tuples

a	b	c_1	D	Sequences of 3-tuples
$2E_1(x)$	$2B_3(x)$	$2x^3 + x^2 - x + 1$	$3x^2 - 3x + 1$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 2x^3 + (4s-3)x^2 + \\ (2s^2 - 4s + 1)x + (-s^2 + 2s) \end{array} \right\}, s = 1,2,3...$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 2s^2x^3 + (-3s^2 + 4s)x^2 + \\ (s^2 - 4s + 2)x + 2s - 1 \end{array} \right\}, s = 1,2,3..$
$2E_1(x)$	$6B_2(x)$	$6x^2 - 2x + 8$	$-12x^3 + 19x^2 + 17$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 6x^2 + (2s^2 + 2s - 6)x \\ + (-s^2 + 8s + 1) \end{array} \right\}, s = 1,2,3...$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 6s^2x^2 + (-6s^2 + 2s + 2)x \\ + s^2 + 8s - 1 \end{array} \right\}, s = 1,2,3..$
$E_0(x)$	$6B_2(x)$	$6x^2 + 2 - 2n$	$3x^2 + (6 - 6n)x + n^2 - 1$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 6x^2 + 6(s-1)x \\ + (s^2 - 2ns + 1) \end{array} \right\}, s = 1,2,3...$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,..}, \left\{ \begin{array}{c} c_s = 6s^2x^2 - (6s^2 - 6s)x \\ - 2sn + (s^2 + 1) \end{array} \right\}, s = 1,2,3..$

$B_0(x)$	$E_2(x)$	$x^2+x+2n+1$	$(2n+1)x+n^2$	$\{a,c_s,c_{s+1}\}_{s=1,2,3,..}$, $\left\{\begin{matrix} c_s = x^2+(2s-1)x \\ +2sn+s^2 \end{matrix}\right\}, s=1,2,3...$ $\{b,c_s,c_{s+1}\}_{s=1,2,3,..}$, $\left\{\begin{matrix} c_s = s^2x^2+(-s^2+2s)x \\ +(2sn+1) \end{matrix}\right\}, s=1,2,3..$
$2B_1(x)$	$4E_3(x)$	$4x^3-4x-2$	$x^4-2x^3-3x^2+4x+2$	$\{a,c_s,c_{s+1}\}_{s=1,2,3,..}$, $\left\{\begin{matrix} c_s = 4x^3+6(s-1)x^2 \\ -(s^2+2s-1)+(2s^2-6s)x \end{matrix}\right\}, s=1,2,3...$ $\{b,c_s,c_{s+1}\}_{s=1,2,3,..}$, $\left\{\begin{matrix} c_s = 4s^2x^3+(6s-6s^2)x^2 \\ +(2-6s)x+(s^2-2s-1) \end{matrix}\right\}, s=1,2,3..$

4. Conclusion:

In this paper, sequences of diophantine triples consisting of Euler and Bernoulli polynomials with suitable properties have been formulated. As numbers have different patterns, one may search for diophantine triples, quadruples and so on for other choices of number patterns with suitable properties.

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