

I and *I*^{*} rough statistical convergence of triple sequences

Ayten Esi^{1,*} and Nagarajan Subramanian²

¹Department of Mathematics, Adiyaman University, 02040, Adiyaman, Turkey

²School of Humanities and Sciences, SASTRA Deemed University, THANJAVUR-613401, India

Received: 5.11.2019

Accepted: 14.12.2019

Published Online: 17.12.2019

Abstract: This paper motivate us to extend the idea of *I*-rough statistical convergence to triple sequences of multiplicity greater than three and also obtain some topological and algebraic properties.

Keywords: triple sequences, rough convergence, admissible ideal, multiple sequences, ideal convergence.

2010 Mathematics Subject Classification. 40F05, 40J05, 40G05.

1. Introduction

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m, n, k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw} . Then the natural density of K is given by $\delta(K) = \lim_{u,v,w \rightarrow \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw} . Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c = \mathbb{N} \setminus K$ is the complement of K . If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, \mathbb{R} denotes the real of three dimensional space with metric (X, d) . Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim x = 0$, provided that the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}| \geq \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x .

If a triple sequence is statistically convergent, then for every $\varepsilon > 0$, infinitely many terms of the sequence may remain outside the ε -neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m, n, k) and we abbreviate this by a.a. (m, n, k) .

* Correspondence: aytenesi@yahoo.com

Let $(x_{m_i n_j k_l})$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \{(m_i, n_j, k_l) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\}$ is different from zero, then $(x_{m_i n_j k_l})$ is called a non thin sub sequence of a triple sequence x .

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \varepsilon\}$$

is different from zero for every $\varepsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x .

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta(\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\}) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [8], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytaç [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [7] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X, ρ) be a metric space. For any non empty closed subsets $A, A_{mnk} \subset X (m, n, k \in \mathbb{N})$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A if the triple sequence $(d(x, A_{mnk}))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_{r,s,t} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x, A_{mnk}) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $St - \lim_{mnk} A_{mnk} = A$ or $A_{mnk} \rightarrow A(WS)$. The triple sequence (A_{mnk}) is bounded if $\sup_{m,n,k} d(x, A_{mnk}) < \infty$ for each $x \in X$.

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where \mathbb{N}, \mathbb{R} and \mathbb{C} denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [9, 10], Esi et al. [2-4], Dutta et al. [6], Subramanian et al. [11], Debnath et al. [5], Aiyub et al. [12] and many others.

Throughout the paper let β be a nonnegative real number.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3 . A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m + n + k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

Definition 1.1 For a triple sequence space $x = (x_{mnk})$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$I - \limsup x = \begin{cases} \sup B_x & , \text{ if } B_x \neq \phi \\ -\infty & , \text{ if } B_x = \phi \end{cases}$$

and

$$I - \liminf x = \begin{cases} \inf A_x & , \text{ if } A_x \neq \phi \\ +\infty & , \text{ if } A_x = \phi \end{cases}$$

where $A_x = \{a \in \mathbb{R} : \{(m, n, k) \in \mathbb{N}^3 : x_{mnk} < a\} \notin I\}$ and $B_x = \{b \in \mathbb{R} : \{(m, n, k) \in \mathbb{N}^3 : x_{mnk} > b\} \notin I\}$.

Definition 1.2 A triple sequence $x = (x_{mnk})$ of real numbers is said to be statistically convergent to $l \in \mathbb{R}^3$ if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - l| \geq \varepsilon\}.$$

Definition 1.3 A triple sequence $x = (x_{mnk})$ of real numbers is said to be statistically convergent to $l \in \mathbb{R}^3$, written as $st - \lim x = l$, provided that the set

$$\{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - l| \geq \varepsilon\},$$

has natural density zero for every $\varepsilon > 0$.

In this case, l is called the statistical limit of the sequence x .

Definition 1.4 Let $x = (x_{mnk})_{m,n,k \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}}$ be a triple sequence in a metric space $(X, |.,. |)$ and r be a non-negative real number. A triple sequence $x = (x_{mnk})$ is said to be r -convergent to $l \in X$, denoted by $x \rightarrow^r l$, if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that for all $m, n, k \geq N_\varepsilon$ we have

$$|x_{mnk} - l| < r + \varepsilon$$

In this case l is called an r -limit of x .

Remark 1.5 We consider r -limit set x which is denoted by LIM_x^r and is defined by

$$LIM_x^r = \{l \in X: x \rightarrow^r l\}.$$

Definition 1.6 A triple sequence $x = (x_{mnk})$ is said to be r -convergent if $LIM_x^r \neq \emptyset$ and r is called a rough convergence degree of x . If $r = 0$ then it is ordinary convergence of triple sequence.

Definition 1.7 Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,. |)$ and r be a non-negative real number is said to be r -statistically convergent to l , denoted by $x \rightarrow^{r-st_3} l$, if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: |x_{mnk} - l| \geq r + \varepsilon\}.$$

In this case l is called r -statistical limit of x . If $r = 0$ then it is ordinary statistical convergent of triple sequence.

Definition 1.8 A class I of subsets of a nonempty set X is said to be an ideal in X provided

- (i) $\emptyset \in I$,
- (ii) $A, B \in I$ implies $A \cup B \in I$.
- (iii) $A \in I, B \subset A$ implies $B \in I$.

I is called a nontrivial ideal if $X \notin I$.

Definition 1.9 A nonempty class F of subsets of a nonempty set X is said to be a filter in X . Provided

- (i) $\emptyset \in F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 1.10 I is a non trivial ideal in $X, X \neq \emptyset$, then the class

$$F(I) = \{M \subset X: M = X \setminus A \text{ for some } A \in I\}$$

is a filter on X , called the filter associated with I .

Definition 1.11 A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Note 1.12 If I is an admissible ideal, then usual convergence in X implies I convergence in X .

Remark 1.13 If I is an admissible ideal, then usual rough convergence implies rough I -convergence.

Definition 1.14 Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |, \cdot, |)$ and r be a non-negative real number is said to be rough ideal convergent or rI -convergent to l , denoted by $x \rightarrow^{rI} l$, if for any $\varepsilon > 0$ we have

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: |x_{mnk} - l| \geq r + \varepsilon\} \in I.$$

In this case l is called rI -limit of x and a triple sequence $x = (x_{mnk})$ is called rough I -convergent to l with r as roughness of degree. If $r = 0$ then it is ordinary I -convergent.

Note 1.15 Generally, a triple sequence $y = (y_{mnk})$ is not I -convergent in usual sense and $|x_{mnk} - y_{mnk}| \leq r$ for all $(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ or

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: |x_{mnk} - y_{mnk}| \geq r\} \in I.$$

for some $r > 0$. Then the triple sequence $x = (x_{mnk})$ is rI -convergent.

Note 1.16 It is clear that rI -limit of x is not necessarily unique.

Definition 1.17 Consider rI -limit set of x , which is denoted by

$$I - LIM_x^r = \{L \in X: x \rightarrow^{rI} l\},$$

then the triple sequence $x = (x_{mnk})$ is said to be rI -convergent if $I - LIM_x^r \neq \emptyset$ and r is called a rough I -convergence degree of x .

Definition 1.18 A triple sequence $x = (x_{mnk}) \in X$ is said to be I -analytic if there exists a positive real number M such that

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: |x_{mnk}|^{1/m+n+k} \geq M\} \in I.$$

Definition 1.19 A point $L \in X$ is said to be an I -accumulation point of a triple sequence $x = (x_{mnk})$ in a metric space (X, d) if and only if for each $\varepsilon > 0$ the set

$$\{(m, n, k) \in \mathbb{N}^3: d(x_{mnk}, l) = |x_{mnk} - l| < \varepsilon\} \notin I.$$

We denote the set of all I -accumulation points of x by $I(\Gamma_x)$.

2. Definitions and Preliminaries

Definition 2.1 Let $I_3 \subset P(\mathbb{N}^3)$ be a non-trivial ideal in \mathbb{N}^3 and β be non-negative real number. A rough triple sequence $x = (x_{mnk})$ is said to be I_3 -rough convergent to some $\bar{0}$, it is denoted by $I_3 - \lim x_{mnk} = \bar{0}$ if for each $\varepsilon > 0$, $\{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - \bar{0}| \geq \beta + \varepsilon\} \in I_3$.

Note We shall denote the set of all I_3 -rough convergent triple sequences by I^3 .

2. I_3 -rough convergence

Theorem 3.1 If a rough triple sequence $x = (x_{mnk})$ is I_3 -rough convergent to some limit, then it must be unique.

Theorem 3.2 Let $x = (x_{mnk})$ and $y = (y_{mnk})$ be two rough triple sequences; then

- (i) If $x = (x_{mnk})$ is rough convergent to $\bar{0}$, then $x = (x_{mnk})$ is I_3 -rough convergent to $\bar{0}$.
- (ii) If $x = (x_{mnk})$ is I_3 -rough convergent to $\bar{0}$ and $c \in \mathbb{R}$, then (cx_{mnk}) is I_3 -rough convergent $c\bar{0}$.
- (iii) If $x = (x_{mnk})$ and $y = (y_{mnk})$ are I_3 -rough convergent to $\bar{0}$ then $(x_{mnk} + y_{mnk})$ is I_3 -rough convergent to $\bar{0}$.

Theorem 3.3 Let $x = (x_{mnk})$ and $y = (y_{mnk})$ be two rough triple sequences, then

- (i) $x_{mnk} \leq y_{mnk}$ for every $(m, n, k) \in K \subset \mathbb{N}^3$ with $K \in I_3$.
- (ii) $I_3 - \lim x_{mnk} = \bar{0}_1$ and $I_3 - \lim y_{mnk} = \bar{0}_2$.
Then $\bar{0}_1 \leq \bar{0}_2$.

Theorem 3.4 Let $x = (x_{mnk})$, $y = (y_{mnk})$ and $z = (z_{mnk})$ be three rough triple sequences such that $x_{mnk} \leq y_{mnk} \leq z_{mnk}$, for every $(m, n, k) \in K$ with $K \in I_3$.

- (ii) $I_3 - \lim x_{mnk} = I_3 - \lim y_{mnk} = I_3 - \lim z_{mnk} = \bar{0}$, then $I_3 - \lim y_{mnk} = \bar{0}$.

Theorem 3.5 Let β be non-negative real number and $I_3 \subset P(\mathbb{N}^3)$ be an admissible ideal in \mathbb{N}^3 . Then the set $I^3 \cap \Lambda^3$ is closed subspace of the metric linear space Λ^3 .

Proof. Let $x^{(rst)} = (x_{mnk}^{(rst)}) \in I^3 \cap \Lambda^3$, and $x^{(rst)} \rightarrow x \in \Lambda^3$. Since $x^{(rst)} \in I^3 \cap \Lambda^3$, therefore there exist (y_{rst}) such that

$$I_3 - \lim_{m,n,k} x_{mnk}^{(rst)} = y_{rst}, \quad r, s, t = 1, 2, 3, \dots \tag{3.1}$$

Furthermore, $x^{(rst)} \rightarrow x$, implies that there exists a positive integer M such that for every $u \geq r \geq M, v \geq b \geq M, w \geq t \geq M$

$$|x^{(uvw)} - x^{(rst)}| < \frac{\beta + \varepsilon}{3}. \tag{3.2}$$

Also, there exist subsets $K_{uvw}, K_{rst} \subset \mathbb{N}^3$ such that $K_{uvw}, K_{rst} \in I_3$ such that

$$\lim_{(m,n,k) \in K_{uvw}} x_{mnk}^{(uvw)} = y_{uvw}, \tag{3.3}$$

$$\lim_{(m,n,k) \in K_{rst}} x_{mnk}^{(rst)} = y_{rst}. \tag{3.4}$$

Now, the set $K_{uvw} \cap K_{rst}$ is non-empty in I_3 . Choose $(x_{ijl}) \in K_{uvw} \cap K_{rst}$, then we have from (3.3) and (3.4)

$$|x_{ijl}^{(uvw)} - y_{uvw}| < \frac{\beta + \varepsilon}{3} \text{ and } |x_{ijl}^{(rst)} - y_{rst}| < \frac{\beta + \varepsilon}{3} \tag{3.5}$$

Hence, for every $u \geq r \geq M, v \geq b \geq M, w \geq t \geq M$, we have from (3.2) to (3.4)

$$|y_{uvw} - y_{rst}| \leq |y_{uvw} - x_{ijl}^{(uvw)}| + |x_{ijl}^{(uvw)} - x_{ijl}^{(rst)}| + |x_{ijl}^{(rst)} - y_{rst}| < \frac{\beta + \varepsilon}{3} + \frac{\beta + \varepsilon}{3} + \frac{\beta + \varepsilon}{3} = \beta + \varepsilon. \tag{3.6}$$

This shows that (y_{rst}) is a rough Cauchy sequence and hence rough convergent. Let,

$$\lim_{r,s,t \rightarrow \infty} y_{rst} = y. \tag{3.7}$$

Next we prove that x is I_3 -rough convergent y . Since $x^{(rst)} \rightarrow x$, by the triple analytic sequence of Λ^3 , it is also coordinatewise convergent. Therefore for each $\varepsilon > 0 \exists$ a positive integer $\eta_1(\beta + \varepsilon)$ such that

$$|x_{mnk}^{(rst)} - x_{mnk}| < \frac{\beta + \varepsilon}{3}; m, n, k \geq \eta_1(\beta + \varepsilon). \tag{3.8}$$

Also from (3.7) we have for every $\varepsilon > 0$ there exists $\eta_2(\beta + \varepsilon)$ such that

$$|y_{mnk} - y| < \frac{\beta + \varepsilon}{3}; m, n, k \geq \eta_2(\beta + \varepsilon). \tag{3.9}$$

Let $\eta_3(\beta + \varepsilon) = \max\{\eta_1(\varepsilon), \eta_2(\varepsilon)\}$ and choose $(r_0 s_0 t_0) > \eta_3(\varepsilon)$. Then for any $(m, n, k) \in \mathbb{N}^3$.

$$\begin{aligned} |x_{mnk} - y| &\leq |x_{mnk}^{(r_0 s_0 t_0)} - x_{mnk}| + |x_{mnk}^{(r_0 s_0 t_0)} - y_{r_0 s_0 t_0}| + |y_{r_0 s_0 t_0} - y| \\ &< \frac{\beta + \varepsilon}{3} + |x_{mnk}^{(r_0 s_0 t_0)} - y_{r_0 s_0 t_0}| + \frac{\beta + \varepsilon}{3} \end{aligned} \tag{3.10}$$

by using (3.8) and (3.9).

Let

$$A_{r_0s_0t_0}\left(\frac{\beta+\varepsilon}{3}\right) = \left\{(m, n, k) \in \mathbb{N}^3: |x_{mnk}^{(r_0s_0t_0)} - y_{r_0s_0t_0}| \geq \frac{\beta+\varepsilon}{3}\right\} \quad (3.11)$$

$$A(\beta + \varepsilon) = \{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - y| \geq \beta + \varepsilon\} \quad (3.12)$$

$$A_{r_0s_0t_0}^c\left(\frac{\beta+\varepsilon}{3}\right) = \left\{(m, n, k) \in \mathbb{N}^3: |x_{mnk}^{(r_0s_0t_0)} - y_{r_0s_0t_0}| < \frac{\beta+\varepsilon}{3}\right\} \quad (3.13)$$

$$A^c(\beta + \varepsilon) = \{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - y| < \beta + \varepsilon\}. \quad (3.14)$$

Hence any $(m, n, k) \in A_{r_0s_0t_0}^c\left(\frac{\beta+\varepsilon}{3}\right)$, we have $|x_{mnk} - y| < \beta + \varepsilon$ and therefore $A_{r_0s_0t_0}^c\left(\frac{\beta+\varepsilon}{3}\right) \subset A^c(\beta + \varepsilon)$. This implies that $A(\beta + \varepsilon) \subset A_{r_0s_0t_0}\left(\frac{\beta+\varepsilon}{3}\right)$. Since $A_{r_0s_0t_0}\left(\frac{\beta+\varepsilon}{3}\right) \in I_3$, we have $A(\beta + \varepsilon) \in I_3$. Hence triple sequence of x is I_3 -rough convergent to triple sequence of y and $x \in I_3$. This shows that $I_3 \cap \Lambda^3$ is closed linear subspace of Λ^3 .

4 I_3 -triple sequence of rough convergence

Definition 4.1 A rough triple sequence $x = (x_{mnk})$ is statistically convergent to $\bar{0}$ if and only if there exists a subset $K = \{m_1n_1k_1 < m_2n_2k_2 < \dots < m_in_jk_l < \dots\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\lim_{m,n,k \rightarrow \infty} x_{mnk} = \bar{0}$.

Definition 4.2 A rough triple sequence $x = (x_{mnk})$ is I_3^* -convergent to $\bar{0}$, if and only if, there exists a subset $K = \{(m_in_jk_l)\} \subset \mathbb{N}^3$, $m, n, k = 1, 2, 3, \dots$ such that $K \in I_3$ and $\lim_{i,j,l \rightarrow \infty} x_{m_in_jk_l} = \bar{0}$.

Theorem 4.3 Let I_3 be an admimissible ideal, β be a non-negative real number and (x_{mnk}) be a rough triple sequence, $I_3^* - \lim x_{mnk} = \bar{0}$ then $I_3 - \lim x_{mnk} = \bar{0}$.

Proof. Let $x = (x_{mnk})$ be a rough triple sequence such that $K = \{(m_in_jk_l)\} \subset \mathbb{N}^3$ such that $K \in I_3$ (i.e $\mathbb{N}^3 - K = H \in I_3$) and $|x_{m_in_jk_l} - \bar{0}| \rightarrow 0$ as $i, j, l \rightarrow \infty$. Then for $\varepsilon > 0$ there exists a positive integer $i_1j_1l_1$ such that $|x_{m_in_jk_l} - \bar{0}| < \beta + \varepsilon$ for all $i > i_1, j > j_1, k > k_1$. Since the set

$$A(\beta + \varepsilon) = \left\{(m_in_jk_l) \in K: |x_{m_in_jk_l} - \bar{0}| \geq \beta + \varepsilon\right\} \subset A \cup H, \quad (4.1)$$

where $A = \{m_1n_1k_1 < m_2n_2k_2 < \dots < m_in_jk_l\}$ and I_3 is admimissible, we have $A \cup H \in I_3$. Hence, we conclude that $I_3 - \lim x_{mnk} = \bar{0}$.

5. I_3 -Cauchy and I_3^* -Cauchy rough triple sequence spaces

Definition 5.1 A rough triple sequence $x = (x_{mnk})$ is said to be I_3 -Cauchy if for each $\varepsilon > 0$ there exists integers $R = R(\beta + \varepsilon)$, $S = S(\beta + \varepsilon)$ and $T = T(\beta + \varepsilon)$ such that $m, p \geq R$, $n, q \geq S$ and $k, r \geq T$,

$$\{(m, n, k) \in \mathbb{N}^3: |x_{mnk} - x_{pqr}| \geq \beta + \varepsilon\} \in I_3. \quad (5.1)$$

Definition 5.2 A rough triple sequence $x = (x_{mnk})$ is said to be I_3^* -Cauchy if there exists a subset $K = \{(m_in_jk_l)\} \subset \mathbb{N}^3$, $i, j, l = 1, 2, 3, \dots$, $K \in I_3$ and the rough triple subsequence $(x_{m_in_jk_l})$ is an ordinary Cauchy sequence.

Theorem 5.3 Let I_3 be an admissible ideal. If a rough triple sequence (x_{mnk}) is I_3^* -Cauchy then it is I_3 -Cauchy.

Theorem 5.4 A rough triple sequence $x = (x_{mnk})$ is I_3 -convergent to $\bar{0}$, if and only if I_3 -Cauchy.

Proof. Let $x = (x_{mnk})$ be I_3 -convergent to for each $\varepsilon \geq 0$ we have

$$A = \left\{ (m, n, k) \in \mathbb{N}^3 : |x_{mnk} - \bar{0}| \geq \frac{\beta + \varepsilon}{2} \right\} \in I_3 \tag{5.2}$$

$$A^c = \left\{ (m, n, k) \in \mathbb{N}^3 : |x_{mnk} - \bar{0}| \leq \frac{\beta + \varepsilon}{2} \right\} \in I_3 \tag{5.3}$$

is non empty. We can choose $(p, q, r) \in A$ such that $|x_{pqr} - \bar{0}| < \beta + \varepsilon$, we denote

$$B = \left\{ (m, n, k) \in \mathbb{N}^3 : |x_{mnk} - x_{pqr}| \geq \frac{\beta + \varepsilon}{2} \right\} \in I_3. \tag{5.4}$$

To Prove that $B \subset A$. Let $(a, b, c) \in B$, then we have

$$(\beta + \varepsilon) < |x_{abc} - x_{pqr}| \leq |x_{abc} - \bar{0}| + |x_{pqr} - \bar{0}| \leq |x_{abc} - \bar{0}| + \frac{\beta + \varepsilon}{2}$$

$$\Rightarrow \frac{\beta + \varepsilon}{2} < |x_{abc} - \bar{0}| \text{ and } (a, b, c) \in A.$$

$$\Rightarrow B \subset A, A \in I_3, \text{ therefore } B \in I_3.$$

Conversely: Suppose that $x = (x_{mnk})$ is a I_3 -Cauchy. To prove that (x_{mnk}) is I_3 -convergent. Let $(\beta + \varepsilon_{pqr} : p, q, r = 1, 2, 3, \dots)$ be strictly decreasing sequence of numbers converges to zero. Since $x = (x_{mnk})$ is I_3 -Cauchy, there exist positive integers of three strictly increasing sequences $(R_{pqr}), (S_{pqr})$ and (T_{pqr}) such that

$$\left\{ (m, n, k) \in \mathbb{N}^3 : |x_{mnk} - x_{R_{pqr}S_{pqr}T_{pqr}}| \geq \frac{\beta + \varepsilon_{pqr}}{2} \right\} \in I_3. \tag{5.5}$$

Clearly, $m_{pqr}n_{pqr}k_{pqr} \in \mathbb{N}^3$ such that

$$|x_{m_{pqr}n_{pqr}k_{pqr}} - x_{R_{pqr}S_{pqr}T_{pqr}}| \leq \beta + \varepsilon_{pqr}. \tag{5.6}$$

It follows that

$$\begin{aligned} |x_{R_{pqr}S_{pqr}T_{pqr}} - x_{R_{pqr}S_{pqr}T_{pqr}}| &\leq |x_{m_{pqr}n_{pqr}k_{pqr}} - x_{R_{pqr}S_{pqr}T_{pqr}}| + |x_{m_{pqr}n_{pqr}k_{pqr}} - x_{R_{pqr}S_{pqr}T_{pqr}}| \\ &\leq (\beta + \varepsilon_{pqr}) + (\beta + \varepsilon_{pqr}) \rightarrow 0 \text{ as } p, q, r \rightarrow \infty. \end{aligned}$$

Thus $\{x_{m_{pqr}n_{pqr}k_{pqr}} : p, q, r = 1, 2, 3, \dots\}$ is Cauchy sequence and satisfies the Cauchy convergence criterion. Let $x_{m_{pqr}n_{pqr}k_{pqr}} \rightarrow \bar{0}$. Since $(\varepsilon_{pqr} : p, q, r = 1, 2, 3, \dots) \rightarrow 0$, given $\varepsilon > 0$, there exists $p_0q_0r_0 \in \mathbb{N}$ such that

$$\beta + \varepsilon_{p_0q_0r_0} < \frac{\beta + \varepsilon}{2} \text{ and } |x_{R_{pqr}S_{pqr}T_{pqr}} - \bar{0}| < \frac{\beta + \varepsilon}{2}, p \geq p_0, q \geq q_0, r \geq r_0. \tag{5.7}$$

Now, we prove that the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - \bar{0}| \geq \beta + \varepsilon\} \subset A_{p_0q_0r_0}. \tag{5.8}$$

Consider arbitrary $(m, n, k) \in \mathbb{N}^3$. By equation (5.7)

$$|x_{mnk} - \bar{0}| \leq |x_{mnk} - x_{R_{pq}S_{pq}T_{pq}}| + |x_{R_{pq}S_{pq}T_{pq}} - \bar{0}| \leq |x_{mnk} - x_{R_{pq}S_{pq}T_{pq}}| + \left(\frac{\beta + \varepsilon}{2}\right)$$

and

$$\beta + \varepsilon_{p_0q_0r_0} < |x_{mnk} - \bar{0}|$$

$\Rightarrow (m, n, k) \in A_{p_0q_0r_0}$ and $A \subset A_{p_0q_0r_0}$, since $A_{p_0q_0r_0} \in I_3$. Hence rough triple sequence of (x_{mnk}) is I_3 -convergent.

Competing Interests: The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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