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Research Article

$k-$Fibonacci and $k-$Lucas Hybrid Numbers

Esra ERKAN [1,*], Ali DAĞDEVİREN [2]

[1] *Department of Mathematics, Faculty of Sciences and Arts, Harran University, Şanlıurfa, Turkey.*
[2] *Turkish Aviation Academy, Weight and Balance Department, Bakırköy, İstanbul, Turkey.*

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Abstract: In this paper, we firstly introduce $k-$Fibonacci and $k-$Lucas hybrid numbers and then give some of their algebraic properties. Additionally, we investigate the Binet formulas, the generating functions, Catalan's, Cassini's and d'Ocagne's identities on these hybrid numbers.

***Keywords**:* Binet Formula, Generating functions, Hybrid numbers, $k-$Fibonacci numbers, $k-$Lucas numbers.

1. INTRODUCTION

A new non-commutative number system which is called hybrid numbers has been defined recently by (Özdemir 2018). One of the important features of this number system is being a generalization of the complex, dual, and hyperbolic (perplex) number systems. In other words, in special cases, these numbers can be obtained. The set of hybrid numbers is denoted by $\mathbb{K}$ and defined by:

$$\mathbb{K} = \left\{ z = a + \mathbf{i}b + \boldsymbol{\varepsilon}c + \mathbf{h}d : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \right\} \tag{1}$$

The Fibonacci sequence is a famous sequence and it has many connections and applications to many branches of mathematics. The $n-$th Fibonacci number F_n is defined by the following recursive relation:

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2,$$

where initial conditions are $F_0 = F_1 = 1$. Therefore, Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ can be given by $\{0,1,1,2,3,5,\ldots\}$. Moreover, different generalizations of Fibonacci sequence have been exhibited in the literature. For these generalizations, we refer to (Bolat 2018; Catarino 2014; Catarino et al. 2014; Falcon et al. 2007; Falcon et al. 2007; Falcon et al. 2007; Godase et al. 2014; Horadam 1961; Larcombe 2013).

Another famous sequence is the Lucas sequence. The $n-$th Lucas number L_n is defined by the following recursive relation

$$L_n = L_{n-1} + L_{n-2} \qquad n \geq 2,$$

where initial conditions are $L_0 = 2$, $L_1 = 1$.The Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$ is given by $\{2,1,3,4,7,\ldots\}$. For more information, see (Hoggat 1969; Koshy 2001; Vajda 1989).

* Correspondence: esraerkan@harran.edu.tr
[1]*ORCID:* 0000-0003-0456-6418
[2]*ORCID: 0000-0003-4887-405X*
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In the literature, there are various studies involving hybrid numbers and sequences with these numbers. For example, Fibonacci and Lucas hybrid numbers, Pell, Jacobstal, and Horadam hybrid numbers were studied in (Catarino 2019; Syznal-Liana et al. 2019; Syznal-Liana et al. 2018; Syznal-Liana et al. 2019; Syznal-Liana et al. 2018).

In this study, after introducing the $k-$Fibonacci and $k-$Lucas hybrid numbers we will give some important properties, the generating functions, and Binet formulas for these numbers. Indeed, we obtain some summing relations between these numbers and some identities such as the Cassini's identity.

2. PRELIMINARIES

Let $\mathbb{K}$ denotes the set of hybrid numbers and $z_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$, $z_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$ be any two hybrid numbers. The addition and subtraction operations are defined as follows:

$$z_1 \pm z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\boldsymbol{\varepsilon} + (d_1 \pm d_1)\mathbf{h}.$$

It is clear that the set of hybrid numbers $\mathbb{K}$ is an Abelian group with the addition operation and also it has the property of associativity. The multiplication of hybrid numbers is defined by the multiplication rules of hybrid units which is given the following table:

Table 1. Multiplication table of hybrid units

$\cdot$	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-\boldsymbol{\varepsilon}-\mathbf{i}$	$\boldsymbol{\varepsilon}$	1

With the help of Table 1, the multiplication of two hybrid numbers are given as follows:

$$\begin{aligned} z_1 z_2 &= (a_1a_2 - b_1b_2 + d_1d_2 + b_1c_2 + c_1b_2) + (a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2)\mathbf{i} \\ &+ (a_1c_2 + b_1d_2 + c_1a_2 - c_1d_2 + d_1b_2 - d_1c_2)\boldsymbol{\varepsilon} + (a_1d_2 - b_1c_2 + c_1b_2 + d_1a_2)\mathbf{h}. \end{aligned}$$

Note that the multiplication of hybrid numbers are not commutative, only associative.

The conjugate of the hybrid number z is indicated by $\overline{z}$ and defined by $\overline{z} = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$. Additionally, the norm of any hybrid number z is defined by as follows:

$$\|z\| = \sqrt{z\overline{z}} = \sqrt{a^2 + (b-c)^2 + c^2 + d^2}.$$

For further information on algebraic and geometric features of hybrid numbers system, see (Özdemir 2018).

Let $\{F_n\}_{n\in\mathbb{N}}$ be a Fibonacci sequence. One of the generalizations of Fibonacci numbers F_n is the $n-$th $k-$ Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$, (Falcon et al. 2007; Falcon et al. 2007). This sequence is given by the following recursive relation:

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2, k \geq 1. \tag{2}$$

The generating function of $n-$th $k-$Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is given as follows:

$$f_k(z) = \frac{z}{1 - kz - z^2}. \tag{3}$$

Moreover, we have the following equalities about the $k-$Fibonacci sequence for $n \geq 0$:

$$F_{k,n}{}^2 + F_{k,n+1}{}^2 = F_{k,2n+1}{}^2, \qquad F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha^n = \alpha F_{k,n} + F_{k,n-1}, \tag{4}$$

where α and β are the roots of the characteristic equation $z^2 - kz - 1 = 0$ associated with the recurrence relation (2) such that

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2}, \qquad \beta = \frac{k - \sqrt{k^2 + 4}}{2}. \tag{5}$$

Additionally, the equation $F_{k,n} = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ is known as Binet's formula and the relations between α and β are given by

$$\alpha + \beta = k, \qquad \alpha - \beta = \sqrt{k^2 + 4}, \qquad \alpha\beta = -1. \tag{6}$$

Let $\{L_n\}_{n\in\mathbb{N}}$ be the Lucas sequence. Like Fibonacci numbers, there is a generalization of Lucas numbers L_n which is called as $n-$th $k-$Lucas sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$. This sequence is

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2},\, n \geq 2,\, k \geq 1. \tag{7}$$

Generating function of $n-$th $k-$Lucas sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$ is given by (Godase et al 2014)

$$l_k(z) = \frac{2 - kz}{1 - kz - z^2}. \tag{8}$$

Furthermore, we have the following equalities:

$$L_{k,n} = F_{k,n-1} + F_{k,n+1}, \quad n \geq 1, \tag{9}$$

$$L_{k,n}{}^2 + L_{k,n+1}{}^2 = (k^2 + 4)F_{k,2n+1}, \tag{10}$$

$$L_{k,n} = \alpha^n + \beta^n, \quad n \geq 0, \quad \text{(Binet's Formula)} \tag{11}$$

where α and β are the roots of the characteristic equation $z^2 - kz - 1 = 0$ associated with the recurrence relation (7). For more information about Lucas numbers and their generalizations, see (Godase et al. 2014; Falcon 2011; Stakhov et al. 2005).

3. $k-$FIBONACCI and $k-$LUCAS HYBRID NUMBERS

In this section, we will define $k-$Fibonacci and $k-$Lucas hybrid numbers and obtain some main results. The $n-$th $k-$Fibonacci hybrid number $\{FH_{k,n}\}$ and $n-$th $k-$Lucas hybrid number $\{LH_{k,n}\}$ can be defined by the definitions of hybrid numbers, $k-$Fibonacci numbers, and $k-$Lucas hybrid numbers, as follows:

$$FH_{k,n} = F_{k,n} + \mathbf{i}F_{k,n+1} + \boldsymbol{\varepsilon}F_{k,n+2} + \mathbf{h}F_{k,n+3}, \tag{12}$$

$$LH_{k,n} = L_{k,n} + \mathbf{i}L_{k,n+1} + \boldsymbol{\varepsilon}L_{k,n+2} + \mathbf{h}L_{k,n+3}, \tag{13}$$

where $F_{k,n}$ and $L_{k,n}$ are $n-$th terms of $k-$Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $k-$Lucas sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$, respectively. They satisfy the undermentioned recursive relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \qquad L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \;\; n \geq 1.$$

Remark 3.1 The sequence $\{FH_{k,n}\}_{n\in\mathbb{N}}$ satisfies the second order recursive relation

$$FH_{k,n} = kFH_{k,n-1} + FH_{k,n-2} \tag{14}$$

with the initial conditions $FH_{k,0} = \mathbf{i} + \boldsymbol{\varepsilon}k + \mathbf{h}\left(k^2+1\right)$ and $FH_{k,1} = 1 + \mathbf{i}k + \boldsymbol{\varepsilon}\left(k^2+1\right) + \mathbf{h}\left(k^3+2k\right)$.

Similarly, the sequence $\left\{LH_{k,n}\right\}_{n\in\mathbb{N}}$ satisfies the second order recursive relation

$$LH_{k,n} = kLH_{k,n-1} + LH_{k,n-2} \tag{15}$$

with the initial conditions $LH_{k,0} = 2 + \mathbf{i}k + \boldsymbol{\varepsilon}\left(k^2+2\right) + \mathbf{h}\left(k^3+3k\right)$ and $LH_{k,1} = k + \mathbf{i}\left(k^2+2\right) + \boldsymbol{\varepsilon}\left(k^3+3k\right) + \mathbf{h}\left(k^4+4k^2+2\right)$.

Now, let $FH^1_{k,n}$ and $FH^2_{k,n}$ be any two $n-$th $k-$Fibonacci hybrid numbers. The addition and subtraction of these $k-$Fibonacci hybrid numbers are defined as follows:

$$\begin{aligned} FH^1_{k,n} \mp FH^2_{k,n} &= \left(F^1_{k,n} + \mathbf{i}F^1_{k,n+1} + \boldsymbol{\varepsilon}F^1_{k,n+2} + \mathbf{h}F^1_{k,n+3}\right) \mp \left(F^2_{k,n} + \mathbf{i}F^2_{k,n+1} + \boldsymbol{\varepsilon}F^2_{k,n+2} + \mathbf{h}F^2_{k,n+3}\right) \\ &= (F^1_{k,n} \mp F^2_{k,n}) + \mathbf{i}(F^1_{k,n+1} \mp F^2_{k,n+1}) + \boldsymbol{\varepsilon}(F^1_{k,n+2} \mp F^2_{k,n+2}) + \mathbf{h}(F^1_{k,n+3} \mp F^2_{k,n+3}). \end{aligned}$$

Furthermore, the multiplication of $k-$Fibonacci hybrid numbers is described by

$$\begin{aligned} FH^1_{k,n}FH^2_{k,n} &= \left(F^1_{k,n} + \mathbf{i}F^1_{k,n+1} + \boldsymbol{\varepsilon}F^1_{k,n+2} + \mathbf{h}F^1_{k,n+3}\right)\left(F^2_{k,n} + \mathbf{i}F^2_{k,n+1} + \boldsymbol{\varepsilon}F^2_{k,n+2} + \mathbf{h}F^2_{k,n+3}\right) \\ &= \left(F^1_{k,n}F^2_{k,n} - F^1_{k,n+1}F^2_{k,n+1} + F^1_{k,n+3}F^2_{k,n+3} + F^1_{k,n+1}F^2_{k,n+2} + F^1_{k,n+2}F^2_{k,n+1}\right) \\ &\quad + \mathbf{i}\left(F^1_{k,n}F^2_{k,n+1} + F^1_{k,n+1}F^2_{k,n} + F^1_{k,n+1}F^2_{k,n+3} - F^1_{k,n+3}F^2_{k,n+1}\right) \\ &\quad + \boldsymbol{\varepsilon}\left(F^1_{k,n}F^2_{k,n+2} + F^1_{k,n+1}F^2_{k,n+3} + F^1_{k,n+2}F^2_{k,n} - F^1_{k,n+2}F^2_{k,n+3} - F^1_{k,n+3}F^2_{k,n+1} + F^1_{k,n+3}F^2_{k,n+2}\right) \\ &\quad + \mathbf{h}\left(F^1_{k,n}F^2_{k,n+3} - F^1_{k,n+1}F^2_{k,n+2} + F^1_{k,n+2}F^2_{k,n+1} + F^1_{k,n+3}F^2_{k,n}\right) \end{aligned}$$

Addition, subtraction and multiplication on $k-$Lucas hybrid numbers can be defined in a similar way.

Definition 3.2 Let $FH_{k,n}$ be an $n-$th $k-$Fibonacci hybrid number. The scalar and vector parts of this $k-$Fibonacci hybrid number are denoted by $S_{FH_{k,n}} = F_{k,n}$ and $V_{FH_{k,n}} = \mathbf{i}F_{k,n+1} + \boldsymbol{\varepsilon}F_{k,n+2} + \mathbf{h}F_{k,n+3}$, respectively. Thus, every $n-$th $k-$Fibonacci hybrid number can be written by

$$FH_{k,n} = S_{FH_{k,n}} + V_{FH_{k,n}}. \tag{16}$$

From the equation (16), the addition and subtraction of any two $k-$Fibonacci hybrid numbers $FH^1_{k,n}$ and $FH^2_{k,n}$ are given by

$$\begin{aligned} FH^1_{k,n} \mp FH^2_{k,n} &= (S_{FH^1_{k,n}} + V_{FH^1_{k,n}}) \mp (S_{FH^2_{k,n}} + V_{FH^2_{k,n}}) \\ &= (S_{FH^1_{k,n}} \mp S_{FH^2_{k,n}}) + (V_{FH^1_{k,n}} \mp V_{FH^2_{k,n}}) \end{aligned}$$

and the multiplication of these $k-$Fibonacci hybrid numbers are given by

$$\begin{aligned} FH^1_{k,n}FH^2_{k,n} &= (S_{FH^1_{k,n}} + V_{FH^1_{k,n}})(S_{FH^2_{k,n}} + V_{FH^2_{k,n}}) \\ &= S_{FH^1_{k,n}}S_{FH^2_{k,n}} - \left\langle V_{FH^1_{k,n}}, V_{FH^2_{k,n}} \right\rangle + S_{FH^1_{k,n}}V_{FH^2_{k,n}} + S_{FH^2_{k,n}}V_{FH^1_{k,n}} + V_{FH^1_{k,n}} \times V_{FH^2_{k,n}}. \end{aligned}$$

Definition 3.3 Let $LH_{k,n}$ be an $n-$th $k-$Lucas hybrid number. The scalar and vector parts of this $k-$Lucas hybrid number are denoted by $S_{LH_{k,n}} = L_{k,n}$ and $V_{LH_{k,n}} = \mathbf{i}L_{k,n+1} + \boldsymbol{\varepsilon}L_{k,n+2} + \mathbf{h}L_{k,n+3}$, respectively. Thus, every $n-$th $k-$Lucas hybrid number $LH_{k,n}$ can be written by

$$LH_{k,n} = S_{LH_{k,n}} + V_{LH_{k,n}}. \tag{17}$$

From (17), the addition and subtraction of any two $n-$ th $k-$ Lucas hybrid numbers $LH_{k,n}^{1}$ and $LH_{k,n}^{2}$ is given by

$$LH_{k,n}^{1} \mp LH_{k,n}^{2} = \left(S_{LH_{k,n}^{1}} + V_{LH_{k,n}^{1}}\right) \mp \left(S_{LH_{k,n}^{2}} + V_{LH_{k,n}^{2}}\right) = \left(S_{LH_{k,n}^{1}} \mp S_{LH_{k,n}^{2}}\right) + \left(V_{LH_{k,n}^{1}} \mp V_{LH_{k,n}^{2}}\right)$$

and the multiplication of these $k-$ Lucas hybrid numbers can be defined by

$$\begin{aligned} LH_{k,n}^{1} LH_{k,n}^{2} &= \left(S_{LH_{k,n}^{1}} + V_{LH_{k,n}^{1}}\right)\left(S_{LH_{k,n}^{2}} + V_{LH_{k,n}^{2}}\right) \\ &= S_{LH_{k,n}^{1}} S_{LH_{k,n}^{2}} - \left\langle V_{LH_{k,n}^{1}}, V_{LH_{k,n}^{2}} \right\rangle + S_{LH_{k,n}^{1}} V_{LH_{k,n}^{2}} + S_{LH_{k,n}^{2}} V_{LH_{k,n}^{1}} + V_{LH_{k,n}^{1}} \times V_{LH_{k,n}^{2}}. \end{aligned}$$

Definition 3.4 The conjugate of $n-$ th $k-$ Fibonacci hybrid number $FH_{k,n}$ is defined by

$$\overline{FH_{k,n}} = F_{k,n} - \mathbf{i}F_{k,n+1} - \boldsymbol{\varepsilon}F_{k,n+2} - \mathbf{h}F_{k,n+3} \tag{18}$$

and the conjugate of $n-$ th $k-$ Lucas hybrid number $LH_{k,n}$ is defined by

$$\overline{LH_{k,n}} = L_{k,n} - \mathbf{i}L_{k,n+1} - \boldsymbol{\varepsilon}L_{k,n+2} - \mathbf{h}L_{k,n+3}. \tag{19}$$

Definition 3.5 The norm of the $n-$ th $k-$ Fibonacci hybrid number $FH_{k,n}$ is defined by

$$\left\|FH_{k,n}\right\|^{2} = F_{k,n}^{2} - k^{2}F_{k,n+2}^{2} - 2(k+1)F_{k,n+1}F_{k,n+2} \tag{20}$$

and the norm of an $n-$ th $k-$ Lucas hybrid number $LH_{k,n}$ is defined by

$$\left\|LH_{k,n}\right\|^{2} = L_{k,n}^{2} - k^{2}L_{k,n+2}^{2} - 2(k+1)L_{k,n+1}L_{k,n+2}. \tag{21}$$

Definition 3.6 The 2×2 matrix representation of the $n-$ th $k-$ Fibonacci hybrid number $FH_{k,n}$ is defined by

$$A_{FH_{k,n}} = \begin{bmatrix} F_{k,n} + F_{k,n+2} & 2F_{k,n+1} + (k-1)F_{k,n+2} \\ (k+1)F_{k,n+2} & F_{k,n} - F_{k,n+2} \end{bmatrix}. \tag{22}$$

There is a bijective map between the set of hybrid number $\mathbb{K}$ and the matrices $\mathbb{M}_{2\times 2}$. The matrix $A_{FH_{k,n}}$ is called *the* $k-$ *Fibonacci hybrid matrix of the* $n-$ th $k-$ *Fibonacci hybrid number* $FH_{k,n}$.

Definition 3.7 The 2×2 matrix representation of the $n-$ th $k-$ Lucas hybrid number $LH_{k,n}$ is defined by

$$A_{LH_{k,n}} = \begin{bmatrix} L_{k,n} + L_{k,n+2} & 2L_{k,n+1} + (k-1)L_{k,n+2} \\ (k+1)L_{k,n+2} & L_{k,n} - L_{k,n+2} \end{bmatrix}. \tag{23}$$

There is also a bijective map between the set of hybrid numbers $\mathbb{K}$ and the matrices $\mathbb{M}_{2\times 2}$ This matrix $A_{LH_{k,n}}$ is called the $k-$Lucas hybrid matrix of the $n-$ th $k-$ Lucas hybrid number $LH_{k,n}$.

Taking into consideration of (20) and (22), we get the following proposition immediately:

Proposition 3.8 If $A_{FH_{k,n}}$ is the $k-$ Fibonacci hybrid matrix of the $n-$ th $k-$ Fibonacci hybrid number $FH_{k,n}$, the following relation satisfies

$$\left\|FH_{k,n}\right\|^{2} = det\left(A_{FH_{k,n}}\right).$$

Taking into consideration of (21) and (23), we get the following propositions immediately:

Proposition 3.9 If $A_{LH_{k,n}}$ is the $k-$Lucas hybrid matrix of the $n-$th $k-$Lucas hybrid number $LH_{k,n}$, the following relation satisfies

$$\left\|LH_{k,n}\right\|^2 = det\left(A_{LH_{k,n}}\right).$$

4. GENERATING FUNCTIONS AND BINET'S FORMULAS FOR $k-$FIBONACCI AND $k-$LUCAS HYBRID NUMBERS

In this section, we will give the generating functions and Binet's formulas for the $k-$Fibonacci hybrid sequence $\left\{FH_{k,n}\right\}_{n\in\mathbb{N}}$ and the $k-$Lucas hybrid sequence $\left\{LH_{k,n}\right\}_{n\in\mathbb{N}}$. Let us consider the generating function $g_{FH_{k,n}}(t)$ which is a power series and its coefficients are the terms of the sequence $\left\{FH_{k,n}\right\}_{n\in\mathbb{N}}$ such that

$$g_{FH_{k,n}}(t) = \sum_{n=0}^{\infty} FH_{k,n}t^n. \tag{23}$$

Similarly, the generating function $g_{LH_{k,n}}(t)$ which is a power series and its coefficients are the terms of the sequence $\left\{LH_{k,n}\right\}_{n\in\mathbb{N}}$ such that

$$g_{LH_{k,n}}(t) = \sum_{n=0}^{\infty} LH_{k,n}t^n. \tag{24}$$

Theorem 4.1 The generating function of the $k-$Fibonacci hybrid sequences $\left\{FH_{k,n}\right\}_{n\in\mathbb{N}}$ is given by

$$g_{FH_{k,n}}(t) = \frac{FH_{k,0} + \left(-kFH_{k,0} + FH_{k,1}\right)t}{1-kt-t^2}.$$

Proof. Using the equation (23) and each terms of the expression $1-kt-t^2$, we obtain following equations:

$$g_{FH_{k,n}}(t) = FH_{k,0} + FH_{k,1}t + FH_{k,2}t^2 + \cdots + FH_{k,n}t^n + \cdots$$

$$-ktg_{FH_{k,n}}(t) = -kFH_{k,0}t - kFH_{k,1}t^2 - kFH_{k,2}t^3 - \cdots - kFH_{k,n}t^{n+1} - \cdots$$

$$-t^2 g_{FH_{k,n}}(t) = -FH_{k,0}t^2 - FH_{k,1}t^3 - FH_{k,2}t^5 - \cdots - FH_{k,n}t^{n+2} - \cdots$$

By summing up these equations and also with using the recurrence relation (14) of $k-$Fibonacci hybrid numbers $\left\{FH_{k,n}\right\}_{n\in\mathbb{N}}$, we have

$$\left(1-kt-t^2\right)g_{FH_{k,n}}(t) = FH_{k,0} + \left(-kFH_{k,0} + FH_{k,1}\right)t$$

and

$$g_{FH_{k,n}}(t) = \frac{FH_{k,0} + \left(-kFH_{k,0} + FH_{k,1}\right)t}{1-kt-t^2}.$$

□

Theorem 4.2 The generating function of the $k-$ Lucas hybrid sequences $\left\{LH_{k,n}\right\}_{n\in\mathbb{N}}$ is given by

$$g_{LH_{k,n}}(t) = \frac{LH_{k,0} + \left(-kLH_{k,0} + LH_{k,1}\right)t}{1-kt-t^2}.$$

Proof. Using the equation (24) and each terms of the expression $1-kt-t^2$, we get the following equations:

$$
\begin{aligned}
g_{LH_{k,n}}(t) &= LH_{k,0} + LH_{k,1}t + LH_{k,2}t^2 + \cdots + LH_{k,n}t^n + \cdots \\
-ktg_{LH_{k,n}}(t) &= -kLH_{k,0}t - kLH_{k,1}t^2 - kLH_{k,2}t^3 - \cdots - kLH_{k,n}t^{n+1} - \cdots \\
-t^2 g_{LH_{k,n}}(t) &= -LH_{k,0}t^2 - LH_{k,1}t^3 - LH_{k,2}t^5 - \cdots - LH_{k,n}t^{n+2} - \cdots
\end{aligned}
$$

By summing up these equations side by side and using the recurrence relation (15) for $k-$Lucas hybrid numbers, we have

$$\left(1-kt-t^2\right) g_{LH_{k,n}}(t) = LH_{k,0} + \left(-kLH_{k,0} + LH_{k,1}\right)t$$

and

$$g_{LH_{k,n}}(t) = \frac{LH_{k,0} + \left(-kLH_{k,0} + LH_{k,1}\right)t}{1-kt-t^2}.$$

□

Theorem 4.3 (Binet's Formula) The $n-$th $k-$Fibonacci hybrid number $FH_{k,n}$ is given by

$$FH_{k,n} = \frac{\hat{r}_1(r_1^n) - \hat{r}_2(r_2^n)}{r_1 - r_2}, \tag{25}$$

where $\hat{r}_1$ and $\hat{r}_2$ are $\hat{r}_1 = 1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1+kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right)$ and $\hat{r}_2 = 1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1+kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right)$, respectively.

Proof. Using the equation (12) and the recursive relation (14) for $FH_{k,n+3}$, we obtain

$$
\begin{aligned}
FH_{k,n} &= F_{k,n} + \mathbf{i}F_{k,n+1} + \boldsymbol{\varepsilon}F_{k,n+2} + \mathbf{h}\left(kF_{k,n+2} + F_{k,n+1}\right) \\
&= F_{k,n} + (\mathbf{i}+\mathbf{h})F_{k,n+1} + (\boldsymbol{\varepsilon} + \mathbf{h}k)F_{k,n+2}.
\end{aligned}
$$

Then, if we use the recursive relation (14) for $FH_{k,n+2}$, then we get

$$
\begin{aligned}
FH_{k,n} &= F_{k,n} + (\mathbf{i}+\mathbf{h})F_{k,n+1} + (\boldsymbol{\varepsilon} + \mathbf{h}k)\left(kF_{k,n+1} + F_{k,n}\right) \\
&= \left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k\right)F_{k,n} + \left(\mathbf{i} + \mathbf{h} + (\boldsymbol{\varepsilon} + \mathbf{h}k)k\right)F_{k,n+1}.
\end{aligned}
$$

Additionally, if we use the Binet's formula for $k-$Fibonacci numbers, then

$$FH_{k,n} = \left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k\right)\frac{r_1^n - r_2^n}{r_1 - r_2} + \left(\mathbf{i} + \mathbf{h} + (\boldsymbol{\varepsilon} + \mathbf{h}k)k\right)\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}.$$

By a straightforward computation, we have

$$
\begin{aligned}
FH_{k,n} &= \frac{r_1^n\left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k + \mathbf{i}r_1 + \boldsymbol{\varepsilon}kr_1 + \mathbf{h}(k^2+1)r_1\right) - r_2^n\left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k + \mathbf{i}r_2 + \boldsymbol{\varepsilon}kr_2 + \mathbf{h}(k^2+1)r_2\right)}{r_1 - r_2} \\
&= \frac{r_1^n\left(1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1+kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right)\right) - r_2^n\left(1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1+kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right)\right)}{r_1 - r_2}.
\end{aligned}
$$

Finally, if we take $1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1+kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right) = \hat{r}_1$ and $1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1+kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right) = \hat{r}_2$ then we get

$$FH_{k,n} = \frac{\hat{r}_1(r_1^n) - \hat{r}_2(r_2^n)}{r_1 - r_2}.$$

□

Theorem 4.4 (Binet's Formula) The $n-$ th $k-$ Lucas hybrid number $LH_{k,n}$ is given by

$$LH_{k,n} = \hat{r}_1(r_1^n) + \hat{r}_2(r_2^n), \tag{26}$$

where $\hat{r}_1$ and $\hat{r}_2$ are $\hat{r}_1 = 1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1 + kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right)$ and $\hat{r}_2 = 1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1 + kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right)$, respectively.

Proof. Using the equation (13) and the recursive relation (15) for $LH_{k,n+3}$, we have

$$LH_{k,n} = L_{k,n} + \mathbf{i}L_{k,n+1} + \boldsymbol{\varepsilon}L_{k,n+2} + \mathbf{h}\left(kL_{k,n+2} + L_{k,n+1}\right) = L_{k,n} + (\mathbf{i} + \mathbf{h})L_{k,n+1} + (\boldsymbol{\varepsilon} + \mathbf{h}k)L_{k,n+2}.$$

Then, if we use the recursive relation (15) for $LH_{k,n+2}$, then we get

$$LH_{k,n} = L_{k,n} + (\mathbf{i} + \mathbf{h})L_{k,n+1} + (\boldsymbol{\varepsilon} + \mathbf{h}k)\left(kL_{k,n+1} + L_{k,n}\right) = \left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k\right)L_{k,n} + \left(\mathbf{i} + \mathbf{h} + (\boldsymbol{\varepsilon} + \mathbf{h}k)k\right)L_{k,n+1}.$$

Additionally, if we use the Binet's formula (11) for $k-$ Lucas numbers, then we get

$$LH_{k,n} = \left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k\right)\left(r_1^n - r_2^n\right) + \left(\mathbf{i} + \mathbf{h} + (\boldsymbol{\varepsilon} + \mathbf{h}k)k\right)\left(r_1^{n+1} - r_2^{n+1}\right).$$

By a straightforward computation, we have

$$\begin{aligned} LH_{k,n} &= r_1^n\left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k + \mathbf{i}r_1 + \boldsymbol{\varepsilon}kr_1 + \mathbf{h}(k^2+1)r_1\right) + r_2^n\left(1 + \boldsymbol{\varepsilon} + \mathbf{h}k + \mathbf{i}r_2 + \boldsymbol{\varepsilon}kr_2 + \mathbf{h}(k^2+1)r_2\right) \\ &= r_1^n\left(1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1 + kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right)\right) + r_2^n\left(1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1 + kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right)\right). \end{aligned}$$

Finally, if we take $1 + \mathbf{i}r_1 + \boldsymbol{\varepsilon}(1 + kr_1) + \mathbf{h}\left(k + (k^2+1)r_1\right) = \hat{r}_1$ and $1 + \mathbf{i}r_2 + \boldsymbol{\varepsilon}(1 + kr_2) + \mathbf{h}\left(k + (k^2+1)r_2\right) = \hat{r}_2$, then we get

$$LH_{k,n} = \hat{r}_1(r_1^n) + \hat{r}_2(r_2^n).$$

□

Proposition 4.5 (Catalan's Identity) Let $n \geq r$ for $n, r \in \mathbb{N}$. Then the following identity holds for $n-$ th $k-$ Fibonacci hybrid number $FH_{k,n}$:

$$FH_{k,n-r}FH_{k,n+r} - \left(FH_{k,n}\right)^2 = (-1)^{n-r}\left(\frac{-s_1r_1^{2r} + s_2r_2^{2r}}{(r_1 - r_2)^2} - (FH_{k,r})^2\right), \tag{27}$$

where $\hat{r}_1$ and $\hat{r}_2$ are as in Theorem 4.3 and also $s_1 = \hat{r}_2\hat{r}_1 - \hat{r}_1^2$ and $s_2 = -\hat{r}_1\hat{r}_2 + \hat{r}_2^2$.

Proof. With using the equation (25) in Theorem 4.3 and considering the reality that the multiplication of two $k-$ Fibonacci hybrid numbers is non-commutative, we obtain

$$\begin{aligned} FH_{k,n-r}FH_{k,n+r} - \left(FH_{k,n}\right)^2 &= \left(\frac{\hat{r}_1(r_1^{n-r}) - \hat{r}_2(r_2^{n-r})}{r_1 - r_2}\right)\left(\frac{\hat{r}_1(r_1^{n+r}) - \hat{r}_2(r_2^{n+r})}{r_1 - r_2}\right) - \left(\frac{\hat{r}_1(r_1^n) - \hat{r}_2(r_2^n)}{r_1 - r_2}\right)^2 \\ &= \frac{(r_1r_2)^n\left(-\hat{r}_1\hat{r}_2\left(\frac{r_2}{r_1}\right)^r + \hat{r}_1\hat{r}_2 - \hat{r}_2\hat{r}_1\left(\frac{r_1}{r_2}\right)^r + \hat{r}_2\hat{r}_1\right)}{(r_1 - r_2)^2}. \end{aligned}$$

Since $r_1r_2 = -1$ for $n-$ th $k-$ Fibonacci hybrid number we get

$$FH_{k,n-r}FH_{k,n+r}-\left(FH_{k,n}\right)^2=(-1)^{n-r}\frac{\left(-\hat{r}_1\hat{r}_2r_2^{2r}-\hat{r}_2\hat{r}_1r_1^{2r}+\hat{r}_1\hat{r}_2(r_1r_2)^r+\hat{r}_2\hat{r}_1(r_1r_2)^r\right)}{(r_1-r_2)^2}.$$

Finally, using the equation (25), $s_1=\hat{r}_2\hat{r}_1-\hat{r}_1^2$ and $s_2=-\hat{r}_1\hat{r}_2+\hat{r}_2^2$, we obtain

$$\begin{aligned}FH_{k,n-r}FH_{k,n+r}-\left(FH_{k,n}\right)^2&=(-1)^{n-r}\left(\frac{-(\hat{r}_2\hat{r}_1-\hat{r}_1^2)r_1^{2r}+(-\hat{r}_1\hat{r}_2+\hat{r}_2^2)r_2^{2r}}{(r_1-r_2)^2}-(FH_{k,r})^2\right)\\&=(-1)^{n-r}\left(\frac{-s_1r_1^{2r}+s_2r_2^{2r}}{(r_1-r_2)^2}-(FH_{k,r})^2\right).\end{aligned}$$

□

Proposition 4.6 (Catalan's Identity) Let $n\geq r$ for $n,r\in\mathbb{N}$. The following relation holds for $n-$th $k-$Lucas hybrid number $LH_{k,n}$:

$$LH_{k,n-r}LH_{k,n+r}-\left(LH_{k,n}\right)^2=(-1)^{n-r}\left(s_1r_1^{2r}+s_2r_2^{2r}-(LH_{k,r})^2\right), \tag{28}$$

where $\hat{r}_1$ and $\hat{r}_2$ are described as in Theorem 4.4 and $s_1=\hat{r}_2\hat{r}_1+\hat{r}_1^2$ and $s_2=\hat{r}_1\hat{r}_2+\hat{r}_2^2$.

Proof. With using the equation (26) in Theorem 4.4 and considering the multiplication of two $k-$Lucas hybrid numbers is non-commutative, we obtain

$$\begin{aligned}LH_{k,n-r}LH_{k,n+r}-\left(LH_{k,n}\right)^2&=\left(\hat{r}_1(r_1^{n-r})+\hat{r}_2(r_2^{n-r})\right)\left(\hat{r}_1(r_1^{n+r})+\hat{r}_2(r_2^{n+r})\right)-\left(\left(\hat{r}_1(r_1^{n})+\hat{r}_2(r_2^{n})\right)\right)^2\\&=\left(r_1r_2\right)^n\left(\hat{r}_1\hat{r}_2\left(\frac{r_2}{r_1}\right)^r+\hat{r}_2\hat{r}_1\left(\frac{r_1}{r_2}\right)^r-\hat{r}_1\hat{r}_2-\hat{r}_2\hat{r}_1\right).\end{aligned}$$

Since $r_1r_2=-1$ for $n-$th $k-$Lucas hybrid numbers, we get

$$LH_{k,n-r}LH_{k,n+r}-\left(LH_{k,n}\right)^2=(-1)^{n-r}\left(\hat{r}_1\hat{r}_2r_2^{2r}+\hat{r}_2\hat{r}_1r_1^{2r}-\hat{r}_1\hat{r}_2\left(r_1r_2\right)^r--\hat{r}_2\hat{r}_1\left(r_1r_2\right)^r\right).$$

Finally, using the equation (26), $s_1=\hat{r}_2\hat{r}_1-\hat{r}_1^2$ and $s_2=-\hat{r}_1\hat{r}_2+\hat{r}_2^2$, we obtain

$$\begin{aligned}LH_{k,n-r}LH_{k,n+r}-\left(LH_{k,n}\right)^2&=(-1)^{n-r}\left((\hat{r}_2\hat{r}_1+\hat{r}_1^2)r_1^{2r}+(\hat{r}_1\hat{r}_2+\hat{r}_2^2)r_2^{2r}-(LH_{k,r})^2\right)\\&=(-1)^{n-r}\left(s_1r_1^{2r}+s_2r_2^{2r}-(LH_{k,r})^2\right).\end{aligned}$$

□

In the following, we give the other identity that is called Cassini's identity which is a special condition of Catalan's identity by taking $r=1$.

Proposition 4.7 (Cassini's Identity) For $n\in\mathbb{N}$ and $k\in\mathbb{Z}^+$, there is the following identity for $n-$th $k-$Fibonacci hybrid number $FH_{k,n}$:

$$FH_{k,n-r}FH_{k,n+r}-\left(FH_{k,n}\right)^2=(-1)^{n-1}\left(\frac{-s_1r_1^2+s_2r_2^2}{(r_1-r_2)^2}-(FH_{k,1})^2\right), \tag{29}$$

where $\hat{r}_1$ and $\hat{r}_2$ are described as in Theorem 4.3, s_1 and s_2 are described as in Proposition 4.5.

Proposition 4.8 (Cassini's Identity) For $n\in\mathbb{N}$ and $k\in\mathbb{Z}^+$, there is the following identity for $n-$th $k-$Lucas hybrid number $LH_{k,n}$

$$LH_{k,n-r}LH_{k,n+r}-\left(LH_{k,n}\right)^2=(-1)^{n-1}\left(s_1r_1^2+s_2r_2^2-(LH_{k,1})^2\right), \tag{30}$$

where $\hat{r}_1$ and $\hat{r}_2$ are described as in Theorem 4.4, s_1 and s_2 are described as in Proposition 4.6.

Finally, we give the d'Ocagne's identity for the $k-$Fibonacci and $k-$Lucas hybrid numbers.

Proposition 4.9 (d'Ocagne's Identity) Let $m\in\mathbb{N}$ and $n\in\mathbb{Z}^+$. Then the following identity for $n-$th $k-$Fibonacci hybrid number $FH_{k,n}$ holds for $m>n$:

$$FH_{k,m}FH_{k,n+1}-FH_{k,m+1}FH_{k,n}=(-1)^n\left[FH_{k,m-n}+\frac{1}{\sqrt{k^2+4}}\left(d_1r_1^{m-n}-d_2r_2^{m-n}\right)\right], \tag{31}$$

where $\hat{r}_1$ and $\hat{r}_2$ are described as in Theorem 4.3 and $d_1=\hat{r}_1\hat{r}_2-\hat{r}_1$, $d_2=\hat{r}_2\hat{r}_1-\hat{r}_2$.

Proof. With using the equation (25) in Theorem 4.3 and considering the non-commutativity of multiplication of two $k-$Fibonacci hybrid numbers, we obtain

$$\begin{aligned}FH_{k,m}FH_{k,n+1}-FH_{k,m+1}FH_{k,n}&=\frac{\hat{r}_1(r_1^m)-\hat{r}_2(r_2^m)}{r_1-r_2}.\frac{\hat{r}_1(r_1^{n+1})-\hat{r}_2(r_2^{n+1})}{r_1-r_2}-\frac{\hat{r}_1(r_1^{m+1})-\hat{r}_2(r_2^{m+1})}{r_1-r_2}.\frac{\hat{r}_1(r_1^n)-\hat{r}_2(r_2^n)}{r_1-r_2}\\&=\frac{\hat{r}_1\hat{r}_2(-r_1^mr_2^{n+1}-r_1^{m+1}r_2^n)-\hat{r}_2\hat{r}_1(r_2^mr_1^{n+1}-r_2^{m+1}r_1^n)}{(r_1-r_2)^2}.\end{aligned}$$

Finally, using the equation (25) for $FH_{k,m-n}$, we obtain

$$FH_{k,m}FH_{k,n+1}-FH_{k,m+1}FH_{k,n}=(-1)^n\left[FH_{k,m-n}+\frac{1}{\sqrt{k^2+4}}\left((\hat{r}_1\hat{r}_2-\hat{r}_1)r_1^{m-n}-(\hat{r}_2\hat{r}_1-\hat{r}_2)r_2^{m-n}\right)\right].$$

If we take $d_1=\hat{r}_1\hat{r}_2-\hat{r}_1$ and $d_2=-\hat{r}_1\hat{r}_2+\hat{r}_2^2$, then we can easily obtain desired result which is given in the equation (31).

□

Proposition 4.10 (d'Ocagne's Identity) Let $m\in\mathbb{N}$ and $n\in\mathbb{Z}^+$. Then the following identity for $n-$th $k-$Lucas hybrid number $LH_{k,n}$ holds for $m>n$:

$$LH_{k,m}LH_{k,n+1}-LH_{k,m+1}LH_{k,n}=(-1)^n\left[LH_{k,m-n}+\left(\frac{\hat{r}_1\hat{r}_2}{\sqrt{k^2+4}}-\hat{r}_1\right)r_1^{m-n}+\left(\frac{\hat{r}_2\hat{r}_1}{\sqrt{k^2+4}}-\hat{r}_2\right)r_2^{m-n}\right], \tag{32}$$

where $\hat{r}_1$ and $\hat{r}_2$ are described as in Theorem 4.4.

Proof. With using the equation (26) in Theorem 4.4 and considering the non-commutativity of multiplication of two $k-$Lucas hybrid numbers, we can get

$$\begin{aligned}LH_{k,m}LH_{k,n+1}-LH_{k,m+1}LH_{k,n}&=\left(\hat{r}_1(r_1^m)+\hat{r}_2(r_2^m)\right).\left(\hat{r}_1(r_1^{n+1})+\hat{r}_2(r_2^{n+1})\right)-\left(\hat{r}_1(r_1^{m+1})+\hat{r}_2(r_2^{m+1})\right).\left(\hat{r}_1(r_1^n)+\hat{r}_2(r_2^n)\right)\\&=\hat{r}_1\hat{r}_2r_1^mr_2^{n+1}+\hat{r}_2\hat{r}_1r_1^{n+1}r_2^m-\hat{r}_1\hat{r}_2r_1^{m+1}r_2^n-\hat{r}_2\hat{r}_1r_1^nr_2^{m+1}.\end{aligned}$$

Also, using the equation (26), then we obtain

$$LH_{k,m}LH_{k,n+1}-LH_{k,m+1}LH_{k,n}=(-1)^n\left[LH_{k,m-n}+\left(\hat{r}_1\hat{r}_2(r_1-r_2)-\hat{r}_1\right)r_1^{m-n}+\left(\hat{r}_2\hat{r}_1(r_1-r_2)-\hat{r}_2\right)r_2^{m-n}\right].$$

Finally, if we take $r_1-r_2=\frac{1}{\sqrt{k^2+4}}$, then we get the equation (31).

□

CONCLUSION

In this work, we defined $k-$Fibonacci and $k-$Lucas hybrid numbers and investigated some of their features. For these numbers, we obtained some formulas and identities what has an important place in the number system, like Binet's Formulas, Cassini's identity. Finally, we gave the other important identities that are the special result of these identities such as Catalan's and d'Ocagne's identities.

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