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Research Article

Estimates of Gamma and Beta Functions

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Abstract: In this paper, we present and describe some of the most important properties of gamma and beta functions. Some new results on the theory of inequalities for these functions are also given.

Keywords*:* Gamma function, Beta function, Inequalities, Estimates.

1. PRELIMINARIES

We present some fundamental relations and identities for gamma and beta functions.

1.1 Gamma function

Let $z \in \mathbb{C} \setminus \mathbb{Z}_{-}$. It is well-known that the classical Euler gamma function can be defined by

$$\Gamma(z) := \begin{cases} \int_0^{+\infty} t^{z-1} e^{-t}\, dt, & if \quad \Re z > 0, \\ -\dfrac{\pi}{z\sin(\pi z)\Gamma(-z)}, & if \quad \Re z < 0, \\ -i\dfrac{\pi}{\sinh(\pi \Im z)\Gamma(1-z)}, & if \quad \Re z = 0, \quad \Im z \neq 0. \end{cases}$$

The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), ... as well as many others. The gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study. This function is an important object in various areas of mathematics. It appears in many different contexts and applications as integration formulas, asymptotic series, hypergeometric series, Riemann zeta function, number theory... Euler's gamma function has also been widely studied (see [1]–[44]). Now, we give some properties of the gamma function (see e.g., [29]).

Recurrence formula:

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

$$\forall n \in \mathbb{N}, \quad \Gamma(n+1) = n!$$
$$\Gamma(1) = \Gamma(2) = 1.$$

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$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Legendre's duplication formula:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{2}$$

Euler's reflection (or complement) formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}.$$

$$\forall y \in \mathbb{R}\setminus\{0\}, \quad \Gamma(iy)\Gamma(1-iy) = -i\frac{\pi}{\sinh(\pi y)}.$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}, \quad z \notin \left(\frac{1}{2}+\mathbb{Z}\right).$$

$$\forall y \in \mathbb{R}, \quad \Gamma\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{1}{2}-iy\right) = \frac{\pi}{\cosh(\pi y)}.$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \quad z \notin \mathbb{Z}.$$

$$\forall y \in \mathbb{R}\setminus\{0\}, \quad \Gamma(iy)\Gamma(-iy) = \frac{\pi}{y\sinh(\pi y)}.$$

$$\forall n \in \mathbb{N}\setminus\{0\}, \quad \Gamma(nz) = \frac{n^{nz-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right).$$

$$\overline{\Gamma(z)} = \Gamma(\overline{z}).$$

$$\forall n \in \mathbb{N}\setminus\{0\},\ \forall y \in \mathbb{R}, \quad |\Gamma(n+1+iy)|^2 = \frac{\pi y}{\sinh(\pi y)}\prod_{k=1}^{n}(k^2+y^2).$$

$$\forall n \in \mathbb{N}\setminus\{0\},\ \forall y \in \mathbb{R}, \quad \left|\Gamma\left(n+\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh(\pi y)}\prod_{k=1}^{n}\left[\left(k-\frac{1}{2}\right)^2+y^2\right].$$

$$\Gamma(x) \sim_{+\infty} \sqrt{2\pi}\, x^{x-\frac{1}{2}}e^{-x}.$$

$$\forall a \in \mathbb{C}, \quad \Gamma(x+a) \sim_{+\infty} \sqrt{2\pi}\, x^{x+a-\frac{1}{2}}e^{-x}.$$

Stirling's (asymptotic) formula:

$$\Gamma(x+1) \sim_{+\infty} \sqrt{2\pi}\, x^{x+\frac{1}{2}}e^{-x}.$$

$$\forall p,q > 0,\ \forall a,b \in \mathbb{C}, \quad \frac{\Gamma(px+a)}{\Gamma(qx+b)} \sim_{+\infty} \frac{p^{a-\frac{1}{2}}}{q^{b-\frac{1}{2}}}x^{(p-q)x+a-b}e^{[p(\ln p-1)-q(\ln q-1)]x}.$$

$$\forall p > 0,\ \forall a,b \in \mathbb{C}, \quad \frac{\Gamma(px+a)}{\Gamma(px+b)} \sim_{+\infty} p^{a-b}x^{a-b}.$$

Wendel's limit:

$$\forall a,b\in\mathbb{C},\quad \frac{\Gamma(x+a)}{\Gamma(x+b)}\sim_{+\infty} x^{a-b}.$$

$$\forall p>0,\ \forall a,b\in\mathbb{C},\quad \frac{\Gamma(px+a)}{\Gamma(x+b)}\sim_{+\infty} p^{a-\frac{1}{2}}x^{(p-1)x+a-b}e^{[p(\ln p-1)+1]x}.$$

$$\forall p,q>0,\ \forall a\in\mathbb{C},\quad \frac{\Gamma(px+a)}{\Gamma(qx+a)}\sim_{+\infty}\left(\frac{p}{q}\right)^{a-\frac{1}{2}}x^{(p-q)x}e^{[p(\ln p-1)-q(\ln q-1)]x}.$$

$$n!\sim_{+\infty}\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}.$$

$$\Gamma(x)\sim_{0^+}\frac{1}{x}.$$

1.2 Beta function

1.2.1 $B(p,q)$, $p,q\in\mathbb{C}\setminus\mathbb{Z}_-$

Let us now consider the useful and related function to the gamma function which occurs in the computation of many definite integrals. It is defined, for $p,q\in\mathbb{C}\setminus\mathbb{Z}_-$, by

Definition 1.2

$$B(p,q):=\begin{cases}\int_0^1 t^{p-1}(1-t)^{q-1}\,dt, & if\quad \Re p,\Re q>0,\\ -\frac{\pi(p+q)\sin(\pi(p+q))}{pq\sin(\pi p)\sin(\pi q)B(-p,-q)}, & if\quad \Re p,\Re q<0,\\ & \quad p,q,p+q\notin\mathbb{Z},\\ -\frac{\pi}{p\sin(\pi p)B(-p,p+q)}, & if\quad \Re p<0<-\Re p<\Re q,\\ & \quad p\notin\mathbb{Z},\\ \frac{(p+q)\sin(\pi(p+q))}{p\sin(\pi p)}B(-(p+q),q), & if\quad \Re p<0<\Re q<-\Re p,\\ & \quad p,p+q\notin\mathbb{Z},\\ -i\frac{\pi}{q\sinh(\pi\Im p)B(1-p,p+q)}, & if\quad \Re p=0<\Re q,\\ & \quad \Im p\neq 0,\\ -i\frac{\pi\sinh(\pi(\Im p+\Im q))}{(1-(p+q))\sinh(\pi\Im p)\sin(\pi\Im q)B(1-p,1-q)}, & if\quad \Re p=\Re q=0,\\ & \quad \Im p,\Im q,\Im p+\Im q\neq 0,\\ i\frac{\pi\sin(\pi(p+q))}{p\sin(\pi p)\sinh(\pi\Im q)B(-p,1-q)}, & if\quad \Re p<0=\Re q,\\ & \quad \Im q\neq 0,\ p,p+q\notin\mathbb{Z},\\ i\frac{\sinh(\pi(\Im p+\Im q))}{\sin(\pi p)}B(1-(p+q),q), & if\quad \Re p<0<\Re q=-\Re p,\\ & \quad p\notin\mathbb{Z},\ \Im p+\Im q\neq 0,\\ 0, & if\quad \Re p\leq 0,\ p\notin\mathbb{Z},\\ & \quad q=-(p+n),\ n\in\mathbb{N},\\ B(q,p). & \end{cases}$$

1.2.2 $B(p,q)$, $p,q>0$

Let $p,q>0$.

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt.$$

We give some properties of the beta function (see [29]).

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

$$B(q,p)=B(p,q).$$

$$B(p,p)=\frac{(\Gamma(p))^2}{\Gamma(2p)}=2^{1-2p}\sqrt{\pi}\,\frac{\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}=2\int_0^{\frac{1}{2}}[t(1-t)]^{p-1}\,dt. \tag{3}$$

There exist many useful forms of the beta integral which can be obtained by an appropriate change of variables. In particular, we have

$$B(p,q)=\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^{p+q}}\,dt. \tag{4}$$

$$0<2^{2(1-p)}<B(p,p),\quad 0<p<1.$$

$$\Gamma\left(p+\frac{1}{2}\right)<\frac{\sqrt{\pi}}{2}\Gamma(p),\quad 0<p<1.$$

$$\Gamma(2p)<\frac{(\Gamma(p))^2}{2^{2(1-p)}},\quad 0<p<1.$$

$$B(1,1)=1.$$

$$0<\frac{1}{2^{2(p-1)}(2p-1)}<B(p,p)<\frac{1}{2^{2(p-1)}},\quad p>1.$$

$$0<\frac{\sqrt{\pi}}{2}\Gamma(p)<\Gamma\left(p+\frac{1}{2}\right)<\frac{(2p-1)\sqrt{\pi}}{2}\Gamma(p),\quad p>1.$$

$$0<2^{2(p-1)}(\Gamma(p))^2<\Gamma(2p)<2^{2(p-1)}(2p-1)(\Gamma(p))^2,\quad p>1.$$

$$B(p,1)=\frac{1}{p}>1,\quad 0<p<1.$$

$$B(1,q)=\frac{1}{q}<1,\quad q>1.$$

$$0<2\ln 2<B(p,1-p)=\frac{\pi}{\sin(\pi p)}<\frac{1}{p(1-p)},\quad 0<p<\frac{1}{2}.$$

$$0<2\,\frac{2^{1-(p+q)}-1}{1-(p+q)}<B(p,q)<\frac{2^{1-(p+q)}(p+q)}{pq},\quad 0<p<\frac{1}{2},\quad 0<p<q<1-p<1.$$

If $0<p<\frac{1}{2}<1-p<q<1$ or $0<1-p\leq\frac{1}{2}\leq p<q<1$, then

$$0<\frac{2^{p+q-1}-1}{2^{p+q-2}(p+q-1)}<B(p,q)<\frac{p+q}{2^{p+q-1}pq}.$$

$$0<\frac{2^{p-1}(2^q q+p+q-1)-q}{2^{p+q-1}q(p+q-1)}<B(p,q)<\frac{2^{q-1}(2^p p+p+q-1)-p}{2^{p+q-1}p(p+q-1)},\quad 0<p<1<q.$$

$$0 < \frac{1}{2^{p+q-2}(p+q-1)} < B(p,q) < \frac{2^{p+q-1}-1}{2^{p+q-2}(p+q-1)}, \quad 1 < p < q.$$

$$B(q,q) \sim_{+\infty} 2\sqrt{\pi}\, 2^{-2q} q^{-\frac{1}{2}} \longrightarrow 0, \quad q \longrightarrow +\infty.$$

$$B(1,q) = \frac{1}{q} \longrightarrow 0, \quad q \longrightarrow +\infty.$$

$$B(p,q) \sim_{+\infty} \Gamma(p) q^{-p} \longrightarrow 0, \quad q \longrightarrow +\infty, \quad p > 0.$$

2. MAIN RESULTS

Let $p \geq q > 0$.

Proposition 2.1

$$B(p,p) \leq B(p,q) \leq B(q,q).$$

Proof. Since $p \geq q$, then on one hand

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \geq \int_0^1 t^{p-1}(1-t)^{p-1}\, dt,$$

on the other hand

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \leq \int_0^1 t^{q-1}(1-t)^{q-1}\, dt.$$

Proposition 2.2

$$\left(\frac{1}{2}\right)^{\max(p-q,1)} B(q,q) \leq B(p,q) \leq \left(\frac{1}{2}\right)^{\min(p-q,1)} B(q,q).$$

Proof. We write

$$\begin{aligned} B(p,q) &= \int_0^{\frac{1}{2}} t^{p-1}(1-t)^{q-1}\, dt + \int_0^{\frac{1}{2}} t^{q-1}(1-t)^{p-1}\, dt \\ &= \int_0^{\frac{1}{2}} [t^{p-q} + (1-t)^{p-q}][t(1-t)]^{q-1}\, dt. \end{aligned}$$

We consider now the function $t \longmapsto t^{p-q} + (1-t)^{p-q}, \quad 0 < t < \frac{1}{2}$. This function is decreasing (resp. increasing) for $p - q \geq 1$ (resp. $0 \leq p - q < 1$). Then

$$\left(\frac{1}{2}\right)^{p-q} B(q,q) \leq B(p,q) \leq \frac{1}{2} B(q,q), \quad for \quad p - q \geq 1,$$

and

$$\frac{1}{2} B(q,q) \leq B(p,q) \leq \left(\frac{1}{2}\right)^{p-q} B(q,q), \quad for \quad 0 \leq p - q < 1.$$

Proposition 2.3

1. If $0 < q < 1$ and $p \geq q$, then $\frac{2^{2(1-q)}}{p-q+1} \leq B(p,q)$.
2. If $p \geq q \geq 1$, then $B(p,q) \leq \frac{1}{2^{2(q-1)}(p-q+1)}$.

Proof. We use the equality $B(p,q) = \int_0^{\frac{1}{2}} [t^{p-q} + (1-t)^{p-q}][t(1-t)]^{q-1}\, dt$, and the inequality $t(1-t) < \frac{1}{4}$, for $0 < t < \frac{1}{2}$. Then in the first case,

$$B(p,q) \geq \left(\frac{1}{4}\right)^{q-1} \int_0^{\frac{1}{2}} \left[t^{p-q} + (1-t)^{p-q}\right] dt = \frac{2^{2(1-q)}}{p-q+1}.$$

And the inequality is reversed in the second case.

Proposition 2.4

1. If $p \geq q \geq 1$, then $\frac{p+q}{2^{p+q-1}pq} \leq B(p,q) \leq \frac{(2^q-1)p+(2^p-1)q}{2^{p+q-1}pq}$.
2. If $0 < q \leq p \leq 1$, then $\frac{(2^q-1)p+(2^p-1)q}{2^{p+q-1}pq} \leq B(p,q) \leq \frac{p+q}{2^{p+q-1}pq}$.
3. If $0 < q \leq 1 \leq p$, then

$$\frac{1}{2^{p+q-1}pq}\left[2^{q-1}q + \max((2^p-1)q, p)\right] \leq B(p,q) \leq \frac{1}{2^{p+q-1}pq}\left[2^{p-1}p + \min((2^q-1)p, q)\right].$$

Proof. We use the equality $B(p,q) = \int_0^{\frac{1}{2}} t^{p-1}(1-t)^{q-1}\,dt + \int_0^{\frac{1}{2}} t^{q-1}(1-t)^{p-1}\,dt$, and the double inequality $\frac{1}{2} < 1-t < 1$, for $0 < t < \frac{1}{2}$.

1. If $p \geq q \geq 1$, then

$$B(p,q) \leq \left(\frac{1}{2}\right)^{p-1} \int_0^{\frac{1}{2}} (1-t)^{q-1}\,dt + \left(\frac{1}{2}\right)^{q-1} \int_0^{\frac{1}{2}} (1-t)^{p-1}\,dt = \frac{(2^q-1)p+(2^p-1)q}{2^{p+q-1}pq},$$

and

$$B(p,q) \geq \left(\frac{1}{2}\right)^{q-1} \int_0^{\frac{1}{2}} t^{p-1}\,dt + \left(\frac{1}{2}\right)^{p-1} \int_0^{\frac{1}{2}} t^{q-1}\,dt = \frac{p+q}{2^{p+q-1}pq}.$$

2. If $0 < q \leq p \leq 1$, then the precedent inequalities are reversed.
3. If $0 < q \leq 1 \leq p$, then

$$B(p,q) \leq \left(\frac{1}{2}\right)^{p-1} \int_0^{\frac{1}{2}} (1-t)^{q-1}\,dt + \int_0^{\frac{1}{2}} t^{q-1}\,dt = \frac{2^q-1}{2^{p+q-1}q} + \frac{1}{2^q q},$$

$$B(p,q) \leq \left(\frac{1}{2}\right)^{q-1} \int_0^{\frac{1}{2}} t^{p-1}\,dt + \int_0^{\frac{1}{2}} t^{q-1}\,dt = \frac{1}{2^{p+q-1}p} + \frac{1}{2^q q},$$

$$B(p,q) \geq \int_0^{\frac{1}{2}} t^{p-1}\,dt + \left(\frac{1}{2}\right)^{p-1} \int_0^{\frac{1}{2}} t^{q-1}\,dt = \frac{1}{2^p p} + \frac{1}{2^{p+q-1}q},$$

and

$$B(p,q) \geq \int_0^{\frac{1}{2}} t^{p-1}\,dt + \left(\frac{1}{2}\right)^{q-1} \int_0^{\frac{1}{2}} (1-t)^{p-1}\,dt = \frac{1}{2^p p} + \frac{2^p-1}{2^{p+q-1}p}.$$

Corollary 2.5

$$\forall p > 0, \quad \frac{\min(2^p-1,1)}{2^{2(p-1)}p} \leq B(p,p) \leq \frac{\max(2^p-1,1)}{2^{2(p-1)}p}.$$

3. APPLICATIONS

Proposition 3.1

$$\forall k > 0, \quad \frac{2^{k+3}}{\pi(2k+1)} \min\left(\left(2^{\frac{k}{2}}-1\right)\left(2^{k+\frac{1}{2}}-1\right), 2^{\frac{k}{2}}-1, 1\right) \le \frac{\sqrt{\pi}\,\Gamma\left(k+\frac{1}{2}\right)}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}$$

$$\le \frac{2^{k+3}}{\pi(2k+1)} \max\left(1, 2^{k+\frac{1}{2}}-1, \left(2^{\frac{k}{2}}-1\right)\left(2^{k+\frac{1}{2}}-1\right)\right).$$

Proof. By using the relations (3), (1), and (2), we get

$$\frac{\sqrt{\pi}\,\Gamma\left(k+\frac{1}{2}\right)}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} = \frac{2^{2(2k-1)}k}{\pi} B\left(\frac{k}{2}, \frac{k}{2}\right) B\left(k+\frac{1}{2}, k+\frac{1}{2}\right).$$

Note that $\frac{k}{2} < k+\frac{1}{2} < 1$, if $0 < k < \frac{1}{2}$, $\frac{k}{2} < 1 \le k+\frac{1}{2}$, if $\frac{1}{2} \le k < 2$, and $1 \le \frac{k}{2} < k+\frac{1}{2}$, if $k \ge 2$. The desired result is now obtained by Corollary 2.5.

Proposition 3.2

If $0 < \theta < \frac{\pi}{2}$ *and* $0 \le \lambda < k$, *then*

$$\frac{\min(2^k-1,1)}{2^{k-1}k}(\sin\theta)^{2k-1} \le 2^{k-1}B(k,k)(\sin\theta)^{2k-1} \le \int_0^\theta (\cos\phi - \cos\theta)^{k-1}\cos(\lambda\phi)\,d\phi$$

$$\le \frac{2^{k-2}B(k,k)B\left(\frac{k+\lambda}{2}, \frac{k-\lambda}{2}\right)}{B(k+\lambda, k-\lambda)}(\sin\theta)^{2k-1}.$$

Proof. Let $0 < \theta < \frac{\pi}{2}$ and $0 \le \lambda < k$. We have

$$\int_0^\theta (\cos\phi - \cos\theta)^{k-1}\cos(\lambda\phi)\,d\phi = \sqrt{\frac{\pi}{2}}\,\Gamma(k)(\sin\theta)^{k-\frac{1}{2}}P_{\lambda-\frac{1}{2}}^{\frac{1}{2}-k}(\cos\theta)$$

$$= \frac{2^{k-1}B(k,k)}{B(k+\lambda,k-\lambda)}(\sin\theta)^{2k-1}\int_0^{+\infty}\frac{t^{k+\lambda-1}}{(1+2t\cos\theta+t^2)^k}\,dt,$$

where $P_{\lambda-\frac{1}{2}}^{\frac{1}{2}-k}(\cos\theta)$ denotes the Legendre function, and

$$\forall t > 0, \quad \frac{1}{(1+t)^{2k}} \le \frac{1}{(1+2t\cos\theta+t^2)^k} \le \frac{1}{(1+t^2)^k}.$$

By using (4), the equality $\int_0^{+\infty}\frac{t^{k+\lambda-1}}{(1+t^2)^k}\,dt = \frac{1}{2}\int_0^{+\infty}\frac{s^{\frac{k+\lambda}{2}-1}}{(1+s)^k}\,ds$, and the first inequality of Corollary 2.5, we get the result.

Corollary 3.3

If $0 < \theta < \frac{\pi}{2}$ *and* $0 \le \lambda < k$, *then*

$$1 \le R_{-(k-\lambda)}^{\left(k-\frac{1}{2},k-\frac{1}{2}\right)}(\cos\theta) = {}_2F_1\left(k+\lambda, k-\lambda; k+\frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\le \frac{B\left(k+\frac{1}{2},\frac{1}{2}\right)}{B\left(\frac{k+\lambda+1}{2},\frac{k-\lambda+1}{2}\right)} \le \frac{\pi(k-\lambda+1)2^{k-\lambda-2-\min(k,1)+\max(\lambda,1)}}{\min\left(2^{\frac{k-\lambda+1}{2}}-1,1\right)},$$

where $R_\mu^{(\alpha,\beta)}(z)$ is the Jacobi function, of index (α,β), normalized by $R_\mu^{(\alpha,\beta)}(1) = 1$.

Proof. We have

$$\int_0^{\theta} (\cos\phi - \cos\theta)^{k-1}\cos(\lambda\phi)\, d\phi = \sqrt{\frac{\pi}{2}}\, \Gamma(k)(\sin\theta)^{k-\frac{1}{2}} P_{\lambda-\frac{1}{2}}^{\frac{1}{2}-k}(\cos\theta),$$

$$P_{\lambda-\frac{1}{2}}^{\frac{1}{2}-k}(\cos\theta) = \tan^{k-\frac{1}{2}}\left(\frac{\theta}{2}\right)\ {}_2F_1\left(\frac{1}{2}-\lambda, \frac{1}{2}+\lambda; k+\frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right),$$

and

$${}_2F_1\left(\frac{1}{2}-\lambda, \frac{1}{2}+\lambda; k+\frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) = \cos^{2k-1}\left(\frac{\theta}{2}\right)\ {}_2F_1\left(k+\lambda, k-\lambda; k+\frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right).$$

Then

$${}_2F_1\left(k+\lambda, k-\lambda; k+\frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{2^k\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k)}(\sin\theta)^{1-2k} \times \int_0^{\theta} (\cos\phi - \cos\theta)^{k-1}\cos(\lambda\phi)\, d\phi.$$

Note that $\frac{2^{2k-1}B(k,k)\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k)} = 1$, and $\frac{B\left(\frac{k+\lambda}{2},\frac{k-\lambda}{2}\right)}{2B(k+\lambda,k-\lambda)} = \frac{B\left(k+\frac{1}{2},\frac{1}{2}\right)}{B\left(\frac{k+\lambda+1}{2},\frac{k-\lambda+1}{2}\right)}$. Now, by using Proposition 3.2, Proposition 2.2 and Corollary 2.5, we finish the proof.

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