A Study of Asymmetric Vibration of Polar Orthotropic Annular Plates Using Boundary Characteristic Orthonormal Polynomials

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Abstract: Classical plate theory is used to study asymmetric vibration of polar orthotropic annular plates of quadratically varying thickness in the radial direction and resting on Winkler elastic foundation. Boundary characteristic orthonormal polynomials are used in Rayleigh-Ritz method. Convergence of the results is tested, and comparisons are made in particular cases with the results already reported in the existing literature. First ten frequencies for variation in orthotropy for all possible four combinations when one boundary is free are shown. Three dimensional mode shapes corresponding to the frequencies are shown.

Keywords: Annular plate, variable thickness, orthotropy, asymmetric vibration, Winkler elastic foundation, Jacobi method.

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1. Introduction

Annular circular plate is the simplest and widely used structural element in various engineering fields. The vibration of such plates has been the subject of various studies. Leissa [1-7] summarized the information in his well-known monograph and six comprehensive review articles. For the orthotropic plates, except for the few cases, no closed form solution exits, and researchers have used different approximation methods. Among them Vijaya Kumar and Ramaiah [8,9], Narita [10,11] and Gutierrez *et al*. [12] used Rayleigh-Ritz method, Greenberg and Stavsky [13] used finite difference method and Ginesu *et al*. [14] and Gorman [15] used finite element method.

A lot of information on annular circular plates having varying thickness is also available in the existing literature. Kim and Dickinson [16] have analyzed composite circular plates as a particular case of annular plates by taking inner radius very small but only few results are given on circular plates and that too for uniform thickness only. Laura et al. [17] have analysed annular circular plates having cylindrical anisotropy and non-uniform thickness using polynomial coordinates functions. Chen [18] studied lateral vibration of isotropic and orthotropic thin annular and circular plates of arbitrarily varying thickness along radius using finite element method and obtained natural frequencies and mode shapes of the axisymmetric and asymmetric modes. In these papers, variation of thickness depends only on one taper parameter and there is no mention of nodal lines and mode shapes. In recent years N. Bhardwaj and his co-researchers [19] studied asymmetric vibration of polar orthotropic annular circular plates of quadratically varying thickness with same boundary conditions

In the present paper, asymmetric vibration of annular plates of polar orthotropic material having quadratically varying thickness (along radius) and resting on Winkler elastic foundation is analysed by using boundary characteristic orthonormal polynomials in Rayleigh-Ritz method. Two taper parameters are used for quadratic thickness variation,

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which give more flexibility to distribute the mass economically according to thickness variation. Many thickness variations can be approximated by it by suitably choosing the values of taper constants. Convergence of frequencies atleast upto five significant figures is shown. Comparison of frequencies in particular cases are made with the results already available in the literature. Mode shapes for first ten normal modes of vibrations for all possible four combinations when one of the boundaries is free are presented. Although a close, agreement is found but present results are found to be better in almost all the cases.

1. ANALYSIS

A thin annular plate of outer radius a, inner radius b variable thickness $h(r)$, made up of orthotropic material and resting on Winkler elastic foundation is considered. The plate is subjected to polar coordinates (r, θ) by taking the center of the plate as origin and the middle plane of the plate in the coordinate plane. For free flexural asymmetric vibration of the plate, let $w(r, \theta, t)$ be the deflection of the plate at any point (r, θ) at any time t.

For applying the Rayleigh-Ritz method, the functional

$$J(W) = \frac{\pi E_r H_a^3}{12 \, s_r} \int_{R_0}^{1} \left[F^3 \left\{ W_{m,RR}^2 + 2e_r v_r W_{m,RR} \left(-\frac{m^2}{R^2} W_m + \frac{1}{R} W_{m,R} \right) + e_r \left(-\frac{m^2}{R^2} W_m + \frac{1}{R} W_{m,R} \right)^2 \right. \right.$$

$$\left. \left. + 4 s_r g_r \left(\frac{m}{R^2} W_m - \frac{m}{R} W_{m,R} \right)^2 \right\} + K_f W_m^2 - \Omega^2 F W_m^2 \right] R \, dR \tag{1}$$

obtained by subtracting the maximum kinetic energy from the maximum strain energy of the plate is to be minimized,

where

$$R = r/a, \quad R_0 = b/a, \quad H_a F(R) = h(r)/a, \quad e_r = E_\theta / E_r = v_\theta / v_r, \quad g_r = G / E_r$$

$$s_r = 12(1 - e_r v_r^2), \quad w(r, \theta, t) = aW(R) \cos \omega T, \quad K_f = s_r a k_f \left(E_r / H_a^3 \right),$$

$$w(R, \theta) = W_m(R) \cos m\theta, \quad T = (t/a)\sqrt{(E_r/\rho)}, \quad \Omega^2 = s_r \omega^2 / H_a^2. \tag{2}$$

Here E_r and E_θ and v_r and v_θ are the Young's modulii and Poission's ratio's in r and θ directions, G is the shear modulus, K_f is the foundation constant, ρ is the density of the plate, H_a is the thickness of the plate at the outer edge $R = 1$, m is the number of nodal diameter and ω is the natural frequency of harmonic vibration. A comma followed by a suffixed variable denotes differentiation with respect to that variable.

For quadratically varying thickness of the plate in the radial direction $F(R)$ is taken as

$$F(R) = 1 - \alpha(1 - R) - \beta(1 - R)^2, \qquad \alpha(1 - R_0) + \beta(1 - R_0)^2 < 1 \tag{3}$$

where α and β are the taper constants.

The N-term approximation of the deflection function is taken as

$$W_m(R) = \sum_{j=1}^{N} c_{mj} \Phi_j(R) , \tag{4}$$

where Φ_j are the orthonormal polynomials satisfying at least the geometric edge conditions of the plate. Using three terms recurrence relation given by Chihara [20], Φ_j are generated as

$$\Phi_j = \varphi_j \Big/ \sqrt{<\varphi_j, \varphi_j>}, \quad \varphi_1 = \left(1 - R^2\right)^p \tag{5}$$

$$\phi_{j+1} = \left(R - \frac{<R\phi_j, \phi_j>}{<\phi_j, \phi_j>}\right)\phi_j - \frac{<\phi_j, \phi_j>}{<\phi_{j-1}, \phi_{j-1}>}\phi_{j-1}, \quad \phi_0 = 0, \tag{6}$$

$$<f, g> = \int_{R_0}^1 F(R)f(R)g(R)\, R\, dR, \tag{7}$$

The values of p is taken 0,1 or 2 according as the plate is subjected to free (*F*), simply supported (*S*) or clamped (*C*) edge conditions.

Substitution of $W_m(R)$ from Equation (4) into energy Equation (1) and then minimization of $J(W)$ as a function of the coefficients c_{mj} leads to the standard eigen value problem:

$$\sum_{j=1}^N \left(a_{ij} - \Omega^2 \delta_{ij}\right)c_{mj} = 0, \qquad\qquad i = 1\,(1)\ N, \tag{8}$$

where
$$\delta_{ij} = \begin{cases} 1 & when \quad i = j \\ 0 & when \quad i \neq j \end{cases}$$

and

$$
\begin{aligned}
a_{ij} = \int_{R_0}^1 \Bigg[F^3 &\Bigg\{ \Phi_{i,RR}\Phi_{j,RR} + e_r \nu_r \Phi_{i,RR}\left(-\frac{m^2}{R^2}\Phi_j + \frac{1}{R}\Phi_{j,R}\right) + e_r \nu_r \Phi_{j,RR}\left(-\frac{m^2}{R^2}\Phi_i + \frac{1}{R}\Phi_{i,R}\right) \\
&+ e_r\left(-\frac{m^2}{R^2}\Phi_i + \frac{1}{R}\Phi_{i,R}\right)\left(-\frac{m^2}{R^2}\Phi_j + \frac{1}{R}\Phi_{j,R}\right) + 4 s_r g_r \left(\frac{m}{R^2}\Phi_i - \frac{m}{R}\Phi_{i,R}\right)\left(\frac{m}{R^2}\Phi_j - \frac{m}{R}\Phi_{j,R}\right) \Bigg\} \\
&+ K_f \Phi_i \Phi_j \Bigg] dR
\end{aligned} \tag{9}
$$

The eigen values (Ω) and the eigen vectors (c_{mj}) are computed by Jacobi method. The mode shapes are computed from Equation (4).

2. RESULTS AND DISCUSSIONS

In all nine parameters $R_0, \alpha\, \beta, K_f, e_r, g_r, p, \nu_r$ and N are used in the analysis of this plate. The values of ν_r and g_r are taken to be 0.31 and 5.0 respectively for all computation except for the Table 2, where other values are also taken for the sake of comparison to known results. The variations in thickness parameters for all possible four combinations are taken for both α and β from –0.4 to 0.4 in steps of 0.1.

Table 1 shows the convergence of first ten frequencies $\Omega_{m,n}$ at least upto five significant figures for all possible four combinations *C-F*, *S-F*, *F-C* and *F-S* of edge conditions at outer and inner edges when $\alpha = \beta = 0.4, e_r = g_r = 5.0, R_0 = 0.5$ and $K_f = 500$. The suffixes m and n with Ω denote number of nodal diameters and number of nodal circles respectively. It can be seen that 7 terms are required to get this accuracy in all the cases.

Comparison of $\Omega_{m,n}$ for polar orthotropic annular plates of uniform thickness with Gorman[15], Narita,[10], and Kim and Dickinson[16] are given in Table 2 and with Gutierrez *et al.* [12] in Table 3. Our results are found to be better even for lesser number of terms in almost all the cases besides agreeing closely with their results.

Table 3 shows the variation in $\Omega_{m,n}$ with increasing e_r for *C-F*, *S-F*, *F-C* and *F-S* plates. It is found that there is no specific pattern in $\Omega_{m,n}$ with the increase of e_r. For *C-F* plate $\Omega_{0,0}, \Omega_{5,0}, \Omega_{0,1}, \Omega_{6,0}$ and $\Omega_{1,1}$ increase whereas $\Omega_{1,0}, \Omega_{2,0}, \Omega_{3,0}, \Omega_{4,0}$ and $\Omega_{2,1}$ decrease. For *S-F* plate $\Omega_{0,0}, \Omega_{3,0}$ and $\Omega_{4,0}$ first increase and then decrease whereas $\Omega_{1,0}, \Omega_{2,0}, \Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{0,1}, \Omega_{5,0}$ and $\Omega_{6,0}$ increase. For *F-C* plate, it is observed that $\Omega_{0,0}$ first increases and then decreases whereas $\Omega_{1,0}, \Omega_{2,0}, \Omega_{3,0}, \Omega_{0,1}, \Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{4,0}, \Omega_{5,0}$ and $\Omega_{6,0}$ increase. For *F-S* plate, $\Omega_{0,1}$ and $\Omega_{3,0}$ first increase and then decrease whereas $\Omega_{1,0}, \Omega_{2,0}, \Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{0,0}, \Omega_{4,0}, \Omega_{5,0}$ and $\Omega_{6,0}$ increase.

Three Dimensional mode shapes for CF and FC boundary conditions are given in Figures 1-2 when when $\alpha = \beta = 0.4$, $e_r = g_r = 5.0$, $R_0 = 0.5$, $K_f = 500$.

3. CONCLUSION

It has been shown that frequencies and three dimensional mode shapes of orthotropic annular circular plate when one of the boundaries is free, with varying thickness resting and on elastic foundation can be solved efficiently by using boundary characteristic orthonormal polynomials in the Rayleigh-Ritz method. It reduces the problem into standard eigenvalue problem. Accuracy of the results can be increased by increasing the order of approximation. But the order of approximation cannot be increased arbitrarily because after a certain order the results start diverging due to accumulation of rounding off errors. This study may help the design engineers before finalizing a design in different engineering applications like aeronautical space shuttle components, machine parts, civil structures, etc. by suitably choosing the values of thickness taper parameters.

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Table 1 : **Convergence of** $\Omega_{m,n}$ **when** $\alpha = \beta = 0.4$, $e_r = g_r = 5.0$, $R_0 = 0.5$, $K_f = 500$

Edge conditions (inner, outer)	N	$\Omega_{0,0}$	$\Omega_{1,0}$	$\Omega_{2,0}$	$\Omega_{3,0}$	$\Omega_{4,0}$	$\Omega_{5,0}$	$\Omega_{0,1}$	$\Omega_{1,1}$	$\Omega_{6,0}$	$\Omega_{2,1}$
C-F	5	26.122	27.122	31.050	40.011	55.348	77.092	79.424	89.337	104.38	114.42
	6	26.122	27.122	31.048	40.002	55.325	77.052	79.424	89.334	104.77	114.40
	7	26.122	27.122	31.047	40.000	55.320	77.044	79.424	89.334	104.76	114.40
	8	26.122	27.122	31.047	40.000	55.320	77.044	79.423	89.334	104.76	114.40

Edge conditions	N	$\Omega_{0,0}$	$\Omega_{1,0}$	$\Omega_{2,0}$	$\Omega_{3,0}$	$\Omega_{4,0}$	$\Omega_{0,1}$	$\Omega_{1,1}$	$\Omega_{5,0}$	$\Omega_{2,0}$	$\Omega_{6,0}$
S-F	4	23.725	24.630	28.444	37.660	23.498	60.194	71.909	75.728	99.841	103.86
	5	23.725	24.630	28.444	37.660	53.496	60.194	71.908	75.720	99.840	103.84
	6	23.725	24.630	28.444	37.660	53.496	60.194	71.908	75.719	99.840	103.83
	7	23.725	24.630	28.444	37.660	53.496	60.194	71.908	75.719	99.840	103.83

Edge conditions	N	$\Omega_{0,0}$	$\Omega_{1,0}$	$\Omega_{2,0}$	$\Omega_{3,0}$	$\Omega_{0,1}$	$\Omega_{1,1}$	$\Omega_{4,0}$	$\Omega_{2,1}$	$\Omega_{5,0}$	$\Omega_{3,1}$
_S-F_	4	30.980	41.688	61.097	85.916	86.360	104.16	117.13	141.73	155.19	184.76
	5	30.980	41.687	61.091	85.898	86.300	104.06	117.09	141.58	155.15	184.61
	6	30.980	41.686	61.089	85.890	86.298	104.06	117.07	141.58	155.08	184.59
	7	30.980	41.686	61.089	85.980	86.298	104.06	117.07	141.58	155.08	184.59

Edge conditions	N	$\Omega_{0,0}$	$\Omega_{1,0}$	$\Omega_{2,0}$	$\Omega_{0,1}$	$\Omega_{3,0}$	$\Omega_{1,1}$	$\Omega_{4,0}$	$\Omega_{2,1}$	$\Omega_{5,0}$	$\Omega_{3,1}$
_F-C_	4	26.620	35.133	52.671	64.449	76.765	83.251	107.55	120.31	145.14	162.45
	5	26.620	35.132	52.667	64.436	76.748	83.221	107.51	120.26	145.06	162.40
	6	26.620	35.132	52.666	64.435	76.744	83.219	107.49	120.25	145.02	162.38
	7	26.620	35.132	52.666	64.435	76.744	83.219	107.49	120.25	145.02	162.38

Table :2   **Comparison of $\Omega_{m,n}$ for polar orthotropic annular plates of uniform thickness when** $e_r = 5.0$, $g_r = 0.356$, $R_0 = 0.5$, $K_f = 0$

Edge conditions (inner,outer)	Spresentce of results	$\Omega_{0,0}$	$\Omega_{1,0}$	$\Omega_{2,0}$	$\Omega_{3,0}$	$\Omega_{0,1}$	$\Omega_{1,1}$	$\Omega_{4,0}$	$\Omega_{2,1}$	$\Omega_{5,0}$	$\Omega_{3,1}$
	Gorman[15]	11.30	18.05	33.80	56.521	73.462	78.439	—	93.794	—	120.08
	Narita[10]	11.30	18.05	—	—	—	—	—	—	—	—
F-C	Kim[16]	11.305	18.045	33.795	56.522	73.459	78.436	85.450	93.790	119.72	120.08
	(Present) 6terms	11.305	18.045	33.795	56.522	73.459	78.436	85.451	93.790	119.72	120.08

Table 3: Comparison of $\Omega_{m,n}$ with ref.[9] for polar orthotropic annular plates of uniform thickness when
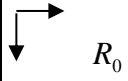
$K_f = 0, v_r = 0.3/e_r, g_r = 1/2(1+v_r)$

Edge conditions (inner, outer)	e_r	R_0 Ref.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
F-S	0.50	[12]	3.835	3.403	3.230	3.244	3.430	3.844	4.658	6.416	11.887
		Present	3.6884	3.3692	3.2257	3.2436	3.4303	3.8443	4.6576	6.4162	11.887
	0.75	[12]	4.434	4.159	4.033	4.086	4.338	4.870	5.9060	8.139	15.082
		Present	4.3585	4.1340	4.0292	4.0854	4.3377	4.8704	5.9060	8.1392	15.082
	1.0	[12]	4.898	4.737	4.668	4.765	5.077	5.711	6.931	9.555	17.709
		Present	4.8533	4.7177	4.6641	4.7640	5.0769	5.7108	6.9310	9.5554	17.709
	1.25	[12]	5.286	5.213	5.201	5.344	5.714	6.439	7.821	10.787	19.993
		Present	5.2572	5.1968	5.1972	5.3434	5.7139	6.4385	7.8209	10.786	19.993
	1.50	[12]	5.625	5.621	5.664	5.855	6.280	7.089	8.618	11.890	22.042
		Present	5.6056	5.6075	5.6610	5.8541	6.2802	7.0886	8.6178	11.890	22.042
F-C	0.50	[12]	9.083	9.167	10.333	12.697	16.966	25.052	42.619	92.591	359.969
		Present	8.6657	9.1308	10.329	12.696	16.966	25.052	42.619	92.591	359.969
	0.75	[12]	9.732	9.858	10.909	13.164	17.347	25.366	42.882	92.813	360.160
		Present	9.6026	9.8258	10.905	13.163	17.347	25.366	42.882	92.813	360.160
	1.0	[12]	10.244	10.437	11.428	13.603	17.715	25.674	43.142	93.035	360.350
		Present	10.159	10.408	11.424	13.603	17.714	25.674	43.142	93.035	360.350
	1.25	[12]	10.577	10.936	11.901	14.019	18.070	25.977	43.400	93.256	360.541
		Present	10.613	10.910	11.896	14.018	18.070	25.977	43.400	93.256	360.541
	1.50	[12]	11.044	11.375	12.334	14.412	18.415	26.275	43.656	93.477	360.731
		Present	11.001	11.351	12.329	14.411	18.415	26.275	43.656	93.476	360.731

Fig. 1: First ten normal modes of vibration of *C-F* plates when $\alpha = \beta = 0.4$, $e_r = g_r = 5.0$, $R_0 = 0.5$, $K_f = 500$.
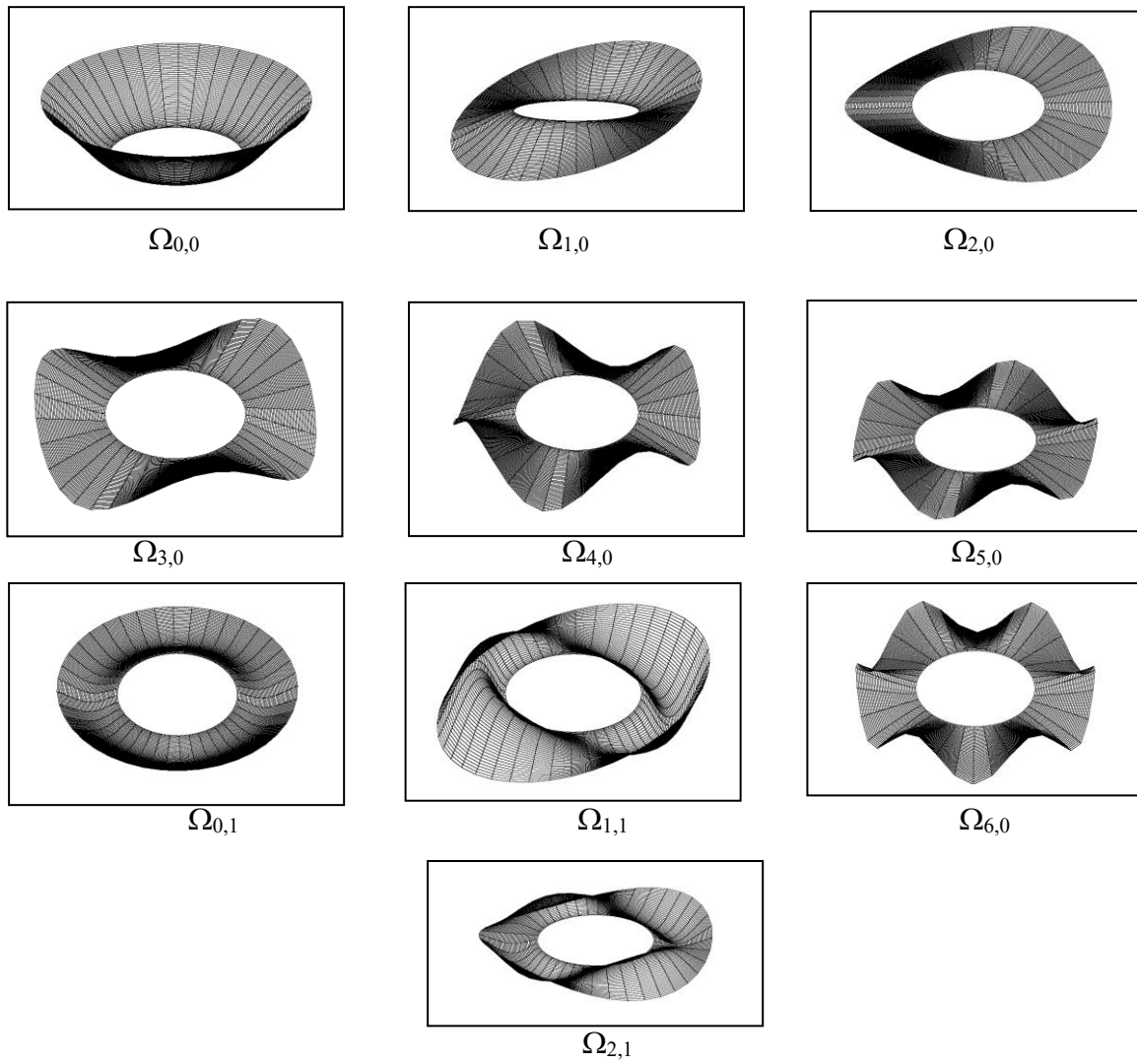

$\Omega_{0,0}$

$\Omega_{1,0}$

$\Omega_{2,0}$

$\Omega_{3,0}$

$\Omega_{4,0}$

$\Omega_{5,0}$

$\Omega_{0,1}$

$\Omega_{1,1}$

$\Omega_{6,0}$

$\Omega_{2,1}$

Fig. 2: First ten normal modes of vibration of *F-C* plates when $\alpha = \beta = 0.4$, $\alpha = \beta = 0.4$, $R_0 = 0.5$, $K_f = 500$.

$\Omega_{0,0}$

$\Omega_{1,0}$

$\Omega_{2,0}$

$\Omega_{3,0}$

$\Omega_{0,1}$

$\Omega_{1,1}$

$\Omega_{4,0}$

$\Omega_{2,1}$

$\Omega_{5,0}$

$\Omega_{3,1}$