Tamap Journal of Mathematics and Statistics
http://www.tamap.org/

doi:10.29371/2019.16.SI01
Volume 2019, Article ID SI01
Research Article

CONSTRUCTION OF SYMMETRIC FUNCTIONS OF GENERALIZED FIBONACCI NUMBERS

Souhila BOUGHABA [1,*], Ali BOUSSAYOUD [2], Mohamed KERADA [3]

[1,2,3] *LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria*

Received: 25.07.2019 • **Accepted**: 31.10.2019 • **Published Online:** 08.11.2019

Abstract: In this paper, we define generalized Fibonacci numbers and a special cases, and we will recover the generating functions of some generalized Fibonacci numbers. The technic used her is based on the theory of the so called symmetric functions.

2010 Mathematics Subject Classification. Primary 05E05; Secondary 11B39.

***Keywords and Phrases*.** Generating functions; Generalized Fibonacci numbers; Symmetric functions.

1. INTRODUCTION

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [12] generalize the Fibonacci sequence $\{F_n\}_{n\in \mathbb{N}}$ by

$$F_n = aF_{n-1} + bF_{n-2},\ n \geq 2$$

with $F_0 = 0$ and $F_1 = 1$.

Horadam [10] defined generalized Fibonacci sequence $\{H_n\}_{n\in \mathbb{N}}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3$$

with $H_1 = p$ and $H_2 = p + q$, where p and q are arbitrary integers.

The generalized Fibonacci sequence $\{U_n\}_{n\in \mathbb{N}}$ is defined by recurrence relation

$$\begin{cases} U_n = aU_{n-1} + bU_{n-2},\ n \geq 2 \\ U_0 = \alpha,\ U_1 = \beta \end{cases},$$

where a, b are arbitrary integers and α, β are complex numbers.

This sequence has been studied by many authors, see for example [10], [12] and [18].

* Correspondence: souhilaboughaba@gmail.com

Note that generalized Fibonacci sequence is the generalization of the well-known sequences k -Fibonacci, k -Lucas, k -Pell, k -Pell Lucas, k -Jacobsthal numbers, Gaussian Fibonacci and Gaussian Lucas numbers, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

In fact, the well-known sequences below are special cases of the generalized Fibonacci sequence

- Putting $a=\beta=k$ and $b=\alpha=1$ reduces to k **-Fibonacci** numbers known as

$$\begin{cases} F_{k,\,n}=kF_{k,\,n-1}+F_{k,\,n-2}, & n\geq 2 \\ F_{k,\,0}=1,\ F_{k,\,1}=k. \end{cases} \tag{1.1}$$

Put $k=1$ in the relationship (1.1) we get Fibonacci numbers $\{F_n\}_{n\in\mathrm{N}}$.

- Substituting $a=\beta=k,\ b=1$ and $\alpha=2$ yields k **-Lucas** numbers given by

$$\begin{cases} L_{k,\,n}=kL_{k,\,n-1}+L_{k,\,n-2}, & n\geq 2 \\ L_{k,\,0}=2, L_{k,\,1}=k. \end{cases} \tag{1.2}$$

Put $k=1$ in the relationship (1.2) we get Lucas numbers $\{L_n\}_{n\in\mathrm{N}}$.

- Taking $a=2,\ b=k,\ \alpha=0$ and $\beta=1$ gives k **-Pell** numbers given by

$$\begin{cases} P_{k,\,n}=2P_{k,\,n-1}+kP_{k,\,n-2}, & n\geq 2 \\ P_{k,\,0}=0,\ P_{k,\,1}=1. \end{cases} \tag{1.3}$$

Put $k=1$ in the relationship (1.3) we get Pell numbers $\{P_n\}_{n\in\mathrm{N}}$.

- Taking $a=\alpha=\beta=2$ and $b=k$ gives k **-Pell Lucas** numbers given by

$$\begin{cases} Q_{k,\,n}=2Q_{k,\,n-1}+kQ_{k,\,n-2}, & n\geq 2 \\ Q_{k,\,0}=2,\ Q_{k,\,1}=2. \end{cases} \tag{1.4}$$

Put $k=1$ in the relationship (1.4) we get Pell Lucas numbers $\{Q_n\}_{n\in\mathrm{N}}$.

- Taking $a=k,\ b=2,\ \alpha=0,\ \beta=1$ gives k **-Jacobsthal** numbers given by

$$\begin{cases} J_{k,\,n}=kJ_{k,\,n-1}+2J_{k,\,n-2}, & n\geq 2 \\ J_{k,\,0}=0,\ J_{k,\,1}=1. \end{cases} \tag{1.5}$$

Put $k=1$ in the relationship (1.5) we get Jacobsthal numbers $\{J_n\}_{n\in\mathrm{N}}$.

- In the case when $a=\beta=1$ and $\alpha=b=2$ and it reduces to **Jacobsthal Lucas** numbers known as

$$\begin{cases} j_n=j_{n-1}+2j_{n-2}, & n\geq 2 \\ j_0=2,\ j_1=1. \end{cases}$$

- In the case when $a=b=\beta=1$ and $\alpha=i$ and it reduces to **Gaussian Fibonacci** numbers known as

$$\begin{cases} GF_n=GF_{n-1}+GF_{n-2}, & n\geq 2 \\ GF_0=i,\ GF_1=1. \end{cases}$$

- In the case when $a=b=1,\ \alpha=2-i$ and $\beta=1+2i$ and it reduces to **Gaussian Lucas** numbers known as

$$\begin{cases} GL_n=GL_{n-1}+GL_{n-2}, & n\geq 2 \\ GL_0=2-i,\ GL_1=1+2i. \end{cases}$$

- In the case when $a=\beta=1,\ b=2$ and $\alpha=\frac{i}{2}$ it reduces to **Gaussian Jacobsthal** numbers known as

$$\begin{cases} GJ_n = GJ_{n-1} + 2GJ_{n-2}, \ n \geq 2 \\ GJ_0 = \frac{i}{2}, \ GJ_1 = 1. \end{cases}$$

- In the case when $a = 1, b = 2, \alpha = 2 - \frac{i}{2}$ and $\beta = 1 + 2i$ it reduces to **Gaussian Jacobsthal Lucas** numbers known as

$$\begin{cases} Gj_n = Gj_{n-1} + 2Gj_{n-2}, \ n \geq 2 \\ Gj_0 = 2 - \frac{i}{2}, \ Gj_1 = 1 + 2i. \end{cases}$$

In order to determine generating functions of generalized Fibonacci numbers, we use analytical means and series manipulation methods. In the sequel, we new symmetric functions and some new properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. DEFINITIONS AND SOME PROPERTIES

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 1: Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_2, ..., a_n)$ is defined by

$$e_k(a_1, a_2, ..., a_n) = \sum_{i_1 + i_2 + ... + i_n = k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (0 \leq k \leq n)$$

with $i_1, i_2, ..., i_n = 0$ or 1 .

Definition 2: Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by

$$h_k(a_1, a_2, ..., a_n) = \sum_{i_1 + i_2 + ... + i_n = k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (0 \leq k \leq n)$$

with $i_1, i_2, ..., i_n \geq 0$.

Remark 1: Set $e_0(a_1, a_2, ..., a_n) = 1$ and $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0,$ we set $e_k(a_1, a_2, ..., a_n) = 0$ and $h_k(a_1, a_2, ..., a_n) = 0$.

Definition 3: [1] Let A and P be any two alphabets. We define $S_n(A-P)$ by the following form

$$\frac{\Pi_{p \varepsilon P}(1-pt)}{\Pi_{a \varepsilon A}(1-at)} = \sum_{n=0}^{\infty} S_n(A-P)t^n, \qquad (2.1)$$

with the condition $S_n(A-P) = 0$ for $n < 0.$

Equation (2.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)t^n = \left(\sum_{n=0}^{\infty} S_n(A)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right), \qquad (2.2)$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P) S_j(A).$$

Definition 4: [5] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1,\cdots,p_i,p_{i+1},\cdots p_n) - f(p_1,\cdots p_{i-1},p_{i+1,}p_i,p_{i+2}\cdots p_n)}{p_i - p_{i+1}}.$$

Definition 5: [2] The symmetrizing operator $\delta^k_{e_1e_2}$ is defined by

$$\delta^k_{p_1p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathrm{N}.$$

3. Generating Function of Generalized Fibonacci Numbers

The following lemmas allows us to obtain many generating functions of generalized Fibonacci numbers and some well-known numbers cited above, using a technique symmetric functions. we refer the reader to see the references [4-12].

Lemma 1: [5] *Given an alphabet* $A = \{a_1, -a_2\}$, *we have*

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 + S_1(-A)z + S_2(-A)z^2}. \qquad (3.1)$$

Lemma 2: [5] *Given an alphabet* $A = \{a_1, -a_2\}$, *we have*

$$\sum_{j=0}^{+\infty} S_{n-1}(a_1 + [-a_2])z^n = \frac{z}{1 + S_1(-A)z + S_2(-A)z^2}. \qquad (3.2)$$

Setting $\begin{cases} S_1(-A) = -a \\ S_2(-A) = -b \end{cases}$ in (3.1) and (3.2) this gives

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 - az - bz^2}. \qquad (3.3)$$

and

$$\sum_{j=0}^{+\infty} S_{n-1}(a_1 + [-a_2])z^n = \frac{z}{1 - az - bz^2}. \qquad (3.4)$$

Multiplying the equation (3.3) by (α) and collect it by the equation (3.4) multiplied by $(\beta - p\alpha)$ we obtain

$$\sum_{n=0}^{\infty} \left[\alpha S_n(a_1 + [-a_2]) + (\beta - p\alpha)S_{n-1}(a_1 + [-a_2])\right]z^n = \frac{\alpha + (\beta - p\alpha)z}{1 - az - bz^2}, \qquad (3.5)$$

and we have the following proposition.

Proposition 1: For $n \in \mathrm{N}$, the new generating function of generalized Fibonacci numbers is given by

$$\sum_{j=0}^{+\infty} U_n t^n = \frac{\alpha + (\beta - p\alpha)z}{1 - az - bz^2},$$

with $U_n = \alpha S_n(a_1 + [-a_2]) + (\beta - p\alpha)S_{n-1}(a_1 + [-a_2])$.

Accordingly, we conclude the following Corollaries.

Corollary 1: *For* $n \in \mathrm{N}$, the generating function of k-Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} F_{k,\,n} t^n = \frac{1}{1-kz-z^2}, \text{ with } F_{k,\,n} = S_n(a_1+[-a_2]). \qquad (3.6)$$

- Put $k=1$ in the relationship (3.6) we have

$$\sum_{n=0}^{\infty} F_n t^n = \frac{1}{1-z-z^2},$$

wich representing a generating function of Fibonacci numbers with $F_n = S_n(a_1+[-a_2])$.

Corollary 2: *For* $n \in \mathrm{N}$, *the generating function of* k *-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{k,\,n} t^n = \frac{2-kz}{1-kz-z^2}, \text{ with } L_{k,\,n} = 2S_n(a_1+[-a_2]) - kS_{n-1}(a_1+[-a_2]). \qquad (3.7)$$

- Put $k=1$ in the relationship (3.7) we have

$$\sum_{n=0}^{\infty} L_n t^n = \frac{2-z}{1-z-z^2},$$

wich representing a generating function of Lucas numbers with $L_n = 2S_n(a_1+[-a_2]) - S_{n-1}(a_1+[-a_2])$.

Corollary 3: *For* $n \in \mathrm{N}$, *the generating function of* k *-Pell numbers is given by*

$$\sum_{n=0}^{\infty} P_{k,\,n} z^n = \frac{z}{1-2z-kz^2}, \text{ with } P_{k,\,n} = S_{n-1}(a_1+[-a_2]). \qquad (3.8)$$

- Put $k=1$ in the relationship (3.8) we have

$$\sum_{n=0}^{\infty} P_n t^n = \frac{z}{1-2z-z^2},$$

wich representing a generating function of Pell numbers with $P_n = S_{n-1}(a_1+[-a_2])$.

Corollary 4: *For* Pell Lucas numbers is given by- k the generating function of $n \in \mathrm{N}$,

$$\sum_{n=0}^{\infty} Q_{k,\,n} z^n = \frac{2-2z}{1-2z-kz^2}, \text{ with } Q_{k,\,n} = 2S_n(a_1+[-a_2]) - 2S_{n-1}(a_1+[-a_2]). \qquad (3.9)$$

- Put $k=1$ in the relationship (3.9) we have

$$\sum_{n=0}^{\infty} Q_n z^n = \frac{2-2z}{1-2z-z^2},$$

Wich representing a generating function of Pell Lucas numbers with:

$$Q_n = 2S_n(a_1+[-a_2]) - 2S_{n-1}(a_1+[-a_2]).$$

Corollary 5: *For* $n \in \mathrm{N}$, the generating function of k -Jacobsthal numbers is given by

$$\sum_{n=0}^{\infty} J_{k,\,n} z^n = \frac{z}{1-kz-z^2}, \text{ with } J_{k,\,n} = S_{n-1}(a_1+[-a_2]). \qquad (3.10)$$

Put $k=1$ in the relationship (3.10) we have

$$\sum_{n=0}^{\infty} J_n z^n = \frac{z}{1-z-z^2},$$

wich representing a generating function of Jacobsthal numbers with $J_n = S_{n-1}(a_1+[-a_2])$.

Corollary 6: *For* $n \in \mathrm{N}$, the generating function of Jacobsthal Lucas numbers is given by

$$\sum_{n=0}^{\infty} j_n z^n = \frac{2-z}{1-z-2z^2}, \text{ with } j_n = 2S_n(a_1+[-a_2]) - S_{n-1}(a_1+[-a_2]).$$

Corollary 7: *For* $n \in \mathrm{N}$, the generating function of Gaussian Jacobsthal numbers is given by

$$\sum_{n=0}^{\infty} GJ_n z^n = \frac{i+(2-i)z}{2-2z-4z^2}, \text{ with } GJ_n = \frac{i}{2} S_n(a_1+[-a_2]) + (1-\frac{i}{2}) S_{n-1}(a_1+[-a_2]).$$

Corollary 8: For $n \in \mathrm{N}$, the generating function of Gaussian Jacobsthal Lucas numbers is given by

$$\sum_{n=0}^{\infty} Gj_n z^n = \frac{4-i+(5i-2)z}{2-2z-4z^2}, \text{ with } Gj_n = (2-\frac{i}{2}) S_n(a_1+[-a_2]) + (\frac{5}{2}i-1) S_{n-1}(a_1+[-a_2]).$$

Corollary 9: For $n \in \mathrm{N}$, the generating function of Gaussian Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} GF_n z^n = \frac{i+(1-i)z}{1-z-z^2}, \text{ with } GF_n = iS_n(a_1+[-a_2]) + (1-i) S_{n-1}(a_1+[-a_2]).$$

Corollary 10: *For* $n \in \mathrm{N}$, the generating function of Gaussian Lucas numbers are given by

$$\sum_{n=0}^{\infty} GL_n z^n = \frac{2-i+(-1+3i)z}{1-z-z^2}, \text{ with } GL_n = (2-i) S_n(a_1+[-a_2]) + (-1+3i) S_{n-1}(a_1+[-a_2]).$$

Conclusion

In this paper, by making use of Eq. (3.1) and (3.2), we have derived some new generating function for generalized Fibonacci numbers. It would be interesting to apply the methods shown in the paper to families of other special numbers.

REFERENCES

[1] D. Kalman and R. Mena, The Fibonacci Numbers--Exposed, The Mathematical Magazine, 2, (2002).

[2] A.F. Horadam, The Generalized Fibonacci Sequences, The American Math. Monthly, 68(5), 455-459, (1961).

[3] V.K. Gupta, Y.K. Panwar, O. Sikhwal, Generalized Fibonacci Sequences, Theoretical Mathematics & Applications, 2(2), 115-124, (2012).

[4] A. Boussayoud, S. Boughaba, On Some Identities and Generating Functions for k-Pell sequences and Chebyshev polynomials, Online J. Anal. Comb. 14 #3, 1-13, (2019).

[5] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, Online.J. Anal. Comb. 12 #1, 1-10,(2017).

[6] A. Boussayoud, A. Abderrezzak, Complete Homogeneous Symmetric Functions and Hadamard Product, Ars Comb. 144, 81-90, (2019).

[7] A. Boussayoud, M. Chelgham, S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, Turkish Journal of Analysis and Number Theory.6(3), 93-97, (2018).

[8] A. Boussayoud, S. Boughaba, M. Kerada, Generating Functions k-Fibonacci and k-Jacobsthal numbers at negative indices, Electron. J. Math. Analysis Appl. 6(2), 195-202, (2018).

[9] A. Boussayoud, M. Bolyer, M. Kerada, On Some Identities and Symmetric Functions for Lucas and Pell numbers, Electron. J. Math. Analysis Appl.5(1), 202-207, (2017).

[10] A. Boussayoud, M. Boulyer, M. Kerada, A simple and accurate method for determination of some generalized sequence of numbers, Int. J. Pure Appl. Math.108, 503-511, (2016).

[11] A. Boussayoud, M. Kerada, A. Abderrezzak, A Generalization of Some Orthogonal Polynomials, Springer Proc. Math. Stat. 41, 229-235, (2013).

[12] S. Boughaba, A. Boussayoud, S. Araci, M. Kerada , Construction of generating function Gaussian-Fibonacci numbers and polynomials, Online J. Anal. Comb. (submitted).

[13] A. Boussayoud, S. Boughaba, M. Kerada, S. Araci, M. Acikgoz, Generating Functions of Binary Products of k-Fibonacci and Orthogonal Polynomials, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.. 113, 1579-1505, 2019.

[14] C. Bolat, H Kose, On the Properties of k-Fibonacci Numbers, Int. J. Contemp. Math. Sciences. 5, 1097-1105, (2010).
[15] G. B. Djordjevic, Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers, Fibonacci Q. 42, 106--113, (2004).
[16] G. B. Djordjevic and H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, Math. Comput.Modelling. 42, 1049--1056, (2005).
[17] Jordan, J. H. Gaussian Fibonacci and Lucas numbers. Fibonacci Quarterly. 3, 315--318, (1965).
[18] Serpil HALICI and Sinan OZ, On Some Gaussian Pell and Pell-Lucas Numbers, Ordu Univ. J. Sci. Tech. 6(1), 8-18, (2016).