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Research Article

# NEW INTEGRAL TRANSFORM "AF TRANSFORM" and SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS

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**Abstract:** In this research paper, a new integral transform, namely AF transform was introduced and applied to solve linear system of Integro-differential equation with constant cofficients. The brilliance of the method in obtaining analytical solution of some system of Volterra integral and Integro differential equation. we will solve some examples by the homotopy perturbation method (HPM) and compare to new integral transform method.

***Keywords***: AF transform, System of Volterra integral and integro-differential equation, constant cofficients, the homotopy perturbation method (HPM).

## 1. Introduction

Recently, a new integral transform was introduced and was named as AF transform which is defined by

$$p(v) = AF\{f(t)\} = v\int_0^\infty e^{-v^2t} f(t)dt \text{ and } AF^{-1}\{v\int_0^\infty e^{-v^2t} f(t)dt\} = f(t) \qquad (1.1)$$

The AF integral transform states that, if f(t) is a piecewise continuous on every finite interval in [0,+∞) and exponential order. The $AF^{-1}$ will be inverse of the AF integral transform (1.1).

**Definition1.1**. The function f(t) is called exponential order on every finite interval in [0,+∞) that satisfying $|f(t)| \le Me^{at}$, $\exists M > 0$ $\forall t \in [0, +\infty)$.

**Theorem 1.2. (Criteria for Convergance).** The AF integral transform AF {f (t)} exists if it has exponential order and integral exists for any b>0.

*Proof.* Since we only need to show convergence for sufficiently large v, assume $v < \sqrt{c}$ *and* $v > 0$. We break the integral of $v\int_0^\infty |f(t)\, e^{-v^2t}\, dt|$ into two integrals, one from 0 to n and another from n to ∞ which we have

$$\begin{aligned} v\int_0^\infty \left|f(t)\, e^{-v^2t}\right|dt &= v\int_0^n \left|f(t)\, e^{-v^2t}\right|dt + v\int_n^\infty \left|f(t)\, e^{-v^2t}\right|dt \\ &\le v\int_0^n |f(t)|dt + v\int_n^\infty e^{-v^2t}\, |f(t)|dt, \end{aligned}$$

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$$\leq \; v\int_0^n |f(t)|dt \; + \; v\int_n^\infty e^{-v^2 t}\, Me^{ct}dt, \qquad\qquad definition\ 1.1$$
$$\leq \; v\int_0^n |f(t)|dt \; + \; vM\frac{e^{(c-v^2)t}}{c-v^2} \;\; where\;\; t = n\; to\; \infty \qquad v > \sqrt{c},$$
$$\leq \; v\int_0^n |f(t)|dt \; + \; vM\frac{e^{(c-v^2)n}}{c-v^2}\;, \qquad\qquad v > \sqrt{c},$$

The first integral exists by assumption, and the second term is finite for $v^2 > c$, so the integral $v\int_0^\infty e^{-v^2 t}\, f(t)dt$ convergence absolutely and the $AF\{f(t)\}$ transform exists■

**Theorem 1.3. (First transfer theorem).** Let P(v) is the $AF\{f(t)\}$ transform function of f(t), i.e. $AF\{f(t)\} = P(v)$. Then

$$AF\{e^{\pm a} f(t)\} = \frac{v}{\sqrt{v^2 \pm a}}\, P\left(\sqrt{v^2 \pm a}\right).$$

*Proof.* It is proved by the following calculation:

$$AF\{e^{\pm a} f(t)\} = \; v\int_0^\infty e^{-v^2 t}\, e^{\pm a}\, f(t)dt = v\int_0^\infty e^{-(v^2 \pm a)t}\, f(t)dt = \frac{v}{\sqrt{v^2 \pm a}}\, P\left(\sqrt{v^2 \pm a}\right) \;■$$

**Theorem 1.4.** Let $AF\{f(t)\} = P(v)$. Then

$$AF\{f^{(n)}(t)\} = v^{(2n)}P(v) - \sum_{k=0}^{n-1} v^{2(n-k)-1}\, f^{(k)}(0), \quad (n \geq 1) \tag{1.2}$$

**Proof:** By substituting $f(t)\; with\; f'(t)$ into (1.1) we obtain $AF\{f'(t)\} = v\int_0^\infty e^{-v^2 t}\, f'(t)dt$. By integrating by part we have

$$AF\{f'(t)\} = v^2 P(v) - vf(0)$$

which equals to (1.2) with n=1 exactly. Let $g(t) = f'(t)$ then $g'(t) = f''(t)$ by substituting $f(t)\; with\; f''(t)$ into (1.1) again we obtain

$$AF\{f''(t)\} = v\int_0^\infty e^{-v^2 t}\, f''(t)dt.$$

By integrating by part we have

$$AF\{f''(t)\} = v^4 P(v) - v^3 f(0) - vf'(0)$$

that this formula is equal to (1.2) with n=2 exactly. (1.2) can be providing by mathematical induction ■

**Theorem 1.5.** Let $AF\{f(t)\} = P(v)$. Then

$$AF\left\{\int_0^t f(w)\, dw\right\} = \frac{1}{v^2} P(v).$$

*Proof.* Let

$$G(t) = \int_0^t f(w)\, dw.$$

Then $G'(t) = f(t)$ and $G(0) = 0$. Taking $AF$ transform of both sides, we have

$$AF\{G'(t)\} = v^2 AF\{G(t)\} - vG(0) = P(v).$$

Then

$$AF\{G(t)\} = \frac{1}{v^2} P(v) \quad \text{and} \quad AF\left\{\int_0^t f(w)\, dw\right\} = \frac{1}{v^2} P(v). \blacksquare$$

**Theorem 1.6.** Let $P(v)$ is the $AF\{f(t)\}$ transform of function $f(t)$ that means $AF\{f(t)\} = P(v)$. Then

$$AF\{tf(t)\} = \frac{-1}{2}\frac{d}{dv}\left(\frac{P(v)}{v}\right) \tag{1.4}$$

$$AF\{t^2 f(t)\} = \left(\frac{-1}{2}\right)^2 \frac{d}{dv}\left(\frac{1}{v}\frac{d}{dv}\left(\frac{P(v)}{v}\right)\right) \tag{1.5}$$

$$AF\{t^n f(t)\} = \left(\frac{-1}{2}\right)^n \frac{d}{dv}\left(\underbrace{\frac{1}{v} \dots \frac{d}{dv}}_{n\ times}\left(\frac{P(v)}{v}\right) \dots\right), \quad (n \geq 1) \tag{1.6}$$

*Proof.* As we know $p(v) = AF\{f(t)\} = v\int_0^\infty e^{-v^2 t} f(t)dt$. First devide both side this equation by $v$ and take derivative respect to v results:

$$\frac{d}{dv}\left(\frac{P(v)}{v}\right) = -2\ \underbrace{v\int_0^\infty e^{-v^2 t}\, tf(t)dt}_{AF\{tf(t)\}}.$$

Then

$$AF\{f(t)\} = \frac{-1}{2}\frac{d}{dv}\left(\frac{P(v)}{v}\right).$$

It follows from (1.4) that

$$AF\{f(t)\} = \frac{-1}{2}\frac{d}{dv}\left(\frac{P(v)}{v}\right).$$

Then

$$v\int_0^\infty e^{-v^2 t}\, tf(t)dt = \frac{-1}{2}\frac{d}{dv}\left(\frac{P(v)}{v}\right).$$

divide both side this equation by $v$ and take derivative respect to $v$ yields

$$AF\{t^2 f(t)\} = \left(\frac{-1}{2}\right)^2 \frac{d}{dv}\left(\frac{1}{v}\frac{d}{dv}\left(\frac{P(v)}{v}\right)\right).$$

Also, (1.6) can be providing by mathematical induction. Therefore, we complete the proof of this theorem. ■

**Theorem 1.7.** Let $P(v)$ and $Q(v)$ are the $AF\{f(x)\}$ and $AF\{g(x)\}$ transform of function $f(x)$f(x) and $g(x)$ that means $AF\{g(x)\} = Q(v)$ and $AF\{f(x)\} = P(v)$. Then the $AF$ transform of the convolution of $f(x)$ and $g(x)$

$$(f * g)(x) = \int_0^\infty f(t)g(t - \tau)d\tau$$

is given by

$$AF\{(f * g)(x)\} = \frac{1}{v}P(v)Q(v). \tag{1.7}$$

*Proof.* The $AF$ transform of $(f * g)(t)$ is defined by

$$AF\{(f * g)(x)\} = v\int_0^\infty e^{-v^2 t} \int_0^\infty f(t)g(t - \tau)d\tau\, dt = v\int_0^\infty f(\tau)d\tau \int_0^\infty e^{-v^2 t}g(t - \tau)\, dt$$

Now setting t − τ = u we have

$$v\int_0^\infty e^{-v^2\tau} f(\tau)d\tau \int_0^\infty e^{-v^2 t}g(t\,)\, dt = \frac{1}{v}\left[\underbrace{v\int_0^\infty e^{-v^2\tau} f(\tau)d\tau}_{P(v)} \quad \underbrace{v\int_0^\infty e^{-v^2\tau} g(\tau)d\tau}_{Q(v)}\right] = \frac{1}{v} P(v)Q(v).$$

Then

$$AF\{(f * g)(x)\} = \frac{1}{v}P(v)Q(v).$$

Thus this complete the proof. ■

## 2. Application to system of Integro-Differential Equations

Let us consider the general first order system of integro-differential equation.

$$\begin{cases} y_1^{(n)}(t) = f(t) + \int_0^t [y_1(x) + y_2(x)]dx \\ y_2^{(n)}(t) = g(t) + \int_0^t [y_2(x) - y_1(x)]dx \end{cases}$$

Whith the initial conditions, $y_1^{(k)}(0) = \alpha_k,\ \ y_2^{(k)}(0) = \beta_k$ . By using $AF$ transform into (2.1) we have

$$\begin{cases} v^{2n}\,\overline{y_1}(v) - \sum_{k=0}^{n-1} v^{2(n-k)-1}\, {y_1}^{(k)}(0) = P(v) + \frac{1}{v^2}\overline{y_1}(v) + \frac{1}{v^2}\overline{y_2}(v) \\ v^{2n}\,\overline{y_2}(v) - \sum_{k=0}^{n-1} v^{2(n-k)-1}\, {y_2}^{(k)}(0) = Q(v) + \frac{1}{v^2}\overline{y_1}(v) - \frac{1}{v^2}\overline{y_2}(v) \end{cases} \tag{2.2}$$

Where $\overline{y_1}$ and $\overline{y_2}$ are $AF$ transform of $y_1, y_2$ respectively. Substituting $y_1^{(k)}(0) = \alpha_k,\ \ y_2^{(k)}(0) = \beta_k$ into (2.2) and solve these equations to find

$$\begin{cases} \overline{y_1}(t) = \frac{v^{2n+2}+1}{v^{4n+4}-2}A(v) + \frac{1}{v^{4n+4}-2}B(v) \\ \overline{y_2}(t) = \frac{1}{v^{4n+4}-2}A(v) + \frac{v^{2n+2}-1}{v^{4n+4}-2}B(v) \end{cases} \tag{2.3}$$

where

$$A(v) = v^2 P(v) + \sum_{k=0}^{n-1} v^{2(n-k)-1}\,\alpha_k \; and \; B(v) = v^2 Q(v) + \sum_{k=0}^{n-1} v^{2(n-k)-1}\,\beta_k.$$

Then

$$y_1(t) = AF^{-1}\left\{\frac{v^{2n+2}+1}{v^{4n+4}-2}A(v) + \frac{1}{v^{4n+4}-2}B(v)\right\}, \qquad y_2(t) = AF^{-1}\left\{\frac{1}{v^{4n+4}-2}A(v) + \frac{v^{2n+2}-1}{v^{4n+4}-2}B(v)\right\}.$$

**Example 2.1**. Consider the following system,

$$\begin{cases} y'_1(t) = t + \int_0^t [y_1(x) + y_2(x)]dx \\ y'_2(t) = \frac{-1}{12}t^4 - 2t + \int_0^t [(t-x)y_1(x)]dx \end{cases} \tag{2.4}$$

with the initial conditions, $y_1(0) = 0$ and $y_2(0) = 1$.

$$\begin{cases} v^2\,\overline{y_1}(v) - v\,y_1(0) = \frac{1}{v^3} + \frac{1}{v^2}\overline{y_1}(v) + \frac{1}{v^2}\overline{y_2}(v) \\ v^2\,\overline{y_2}(v) - v\,y_2(0) = \frac{-1}{12}\frac{4!}{v^9} - \frac{2}{v^3} + \frac{1}{v^4}\overline{y_1}(v) \end{cases} \tag{2.5}$$

$$\begin{cases} v^{11}\,\overline{y_2}(v) - v^5\,\overline{y_1}(v) = v^{10} - 2v^6 - 2 \\ v\,\overline{y_2}(v) + (v^5 - v)\overline{y_1}(v) = 1 \end{cases} \tag{2.6}$$

The solution of these equations is

$$\overline{y_1}(v) = \frac{2}{v^5}, \quad y_1(t) = t^2, \qquad \overline{y_2}(v) = \frac{1}{v} - \frac{2}{v^5} \quad \text{and} \quad y_2(t) = 1 - t^2.$$

**Example 2.2**. Consider the following system,

$$\begin{cases} y''_1(t) = -1 - t^2 - \sin t + \int_0^t [y_1(x) + y_2(x)]dx \\ y''_2(t) = -1 - 2\sin t - \cos t + \int_0^t [y_1(x) - y_2(x)]dx \end{cases} \tag{2.7}$$

subjected to the initial conditions: $y_1(0) = 1$, $y'_1(0) = 1$, $y_2(0) = 0$, $y'_2(0) = 2$. Applying the *AF* transform to both equations (2.7), the result is as follows:

$$\begin{cases} (v^6 - 1)\,\overline{y_1}(v) - \overline{y_2}(v) = v^5 + v^3 - v - \frac{2}{v^3} - \frac{v^3}{v^4+1} \\ -\overline{y_1}(v) + (v^6 + 1)\,\overline{y_2}(v) = 2v^3 - v - \frac{2v^3}{v^4+1} - \frac{2}{v^3} - \frac{v^5}{v^4+1} \end{cases} \tag{2.8}$$

According to (2.8), after some simplication and substitution, the following sets of relations are resulted:

$$y_1(t) = t + \cos t, \qquad y_2(t) = t + \sin t.$$

Now, we will solve this example by the homotopy perturbation method (HPM) [3]. To do this, we construct a homotopy function as the following form:

$$\begin{cases} \mathrm{H}(y_1(t); p) = y_1(t) + 1 + t^2 + \sin t - P\int_0^t [y_1(x) + y_2(x)]dx \\ \mathrm{H}(y_2(t); p) = y_2(t) + 1 + 2\sin t + 2\cos t - P\int_0^t [y_1(x) - y_2(x)]dx \end{cases} \tag{2.9}$$

The embedding parameter p monotonically increases from 0 to 1. In order to apply this method the following expansion will be used

$$y_1(t) = \sum_{n=0}^{\infty} P^n\, y_{1n}(t), \qquad y_2(t) = \sum_{n=0}^{\infty} P^n\, y_{2n}(t), \tag{2.10}$$

where $y_{1n}(t)$ and $y_{2n}(t), n \geq 0$ are the components of $y_1(t)$ *and* $y_2(t)$ that will be elegantly determined in the recursive manner. Substituting (2.10), in (2.9), and equating the terms with equal powers, the following sets of relations are resulted:

$$y_{10}(t) = 1 - \frac{t^2}{2} - \frac{t^4}{12} + \sin t,$$

$$y_{11}(t) = -3 + t + \frac{3t^2}{2} - \frac{t^7}{2520} + 3\cos t - \sin t,$$

$$y_{12}(t) = 2t - \frac{t^3}{3} - \frac{t^5}{60} - \frac{t^5}{60} + \frac{t^6}{360} - \frac{t^8}{20160} - \frac{t^{10}}{907200} - 2\sin t,$$

$$y_{13}(t) = 6 - 2t - 3t^2 + \frac{t^3}{3} + \frac{t^4}{4} - \frac{t^5}{60} - \frac{t^6}{120} + \frac{t^7}{2520} + \frac{t^8}{6720} - \frac{t^{13}}{1556755200} - 6\cos t + 2\sin t,$$

$$y_{20}(t) = -1 + \frac{t^2}{2} + 2\sin t + \cos t,$$

$$y_{21}(t) = 1 - t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^5}{60} - \frac{t^7}{2520} + \sin t - \cos t,$$

$$y_{22}(t) = 2 + 4t - t^2 - \frac{2t^3}{3} + \frac{t^4}{12} + \frac{t^5}{30} - \frac{t^6}{360} + \frac{t^8}{20160} - 4\sin t - 2\cos t,$$

$$y_{23}(t) = -2 + 2t + t^2 - \frac{t^3}{3} - \frac{t^4}{12} + \frac{t^5}{60} + \frac{t^6}{360} - \frac{t^7}{2520} - \frac{t^{11}}{9979200} - \frac{t^{13}}{1556755200} + 2cos\, t - 2\sin t,$$

and so on. Therefore, the solutions by the HPM with three terms will be determined as:

$$\begin{cases} y_1(t) = 4 + t - 2t^2 + \frac{t^4}{6} - \frac{t^6}{180} + \frac{t^8}{10080} - \frac{t^{10}}{907200} - \frac{t^{13}}{1556755200} - 3cos\, t, \\ y_2(t) = 5t - \frac{2t^3}{3} + \frac{t^5}{30} - \frac{t^7}{1260} + \frac{t^9}{90720} - \frac{t^{11}}{9979200} - \frac{t^{13}}{1556755200} - 3sin\, t. \end{cases} \tag{2.11}$$

We substituting cost and sint with Taylor series in (2.11), the following sets of relations are resulted

$$y_1(t) = t + \cos t \ \text{ and } \ y_2(t) = t + \sin t$$

Equal to solutions of (2.8) exactly.

## 3. Conclusion

In this work, we have applied the Reconstruction of Variational Iteration Method AF transform for solving the systems of Volterra integro-differential equations. In our method knowing the variational theory is not essential while it was needed in the variational iteration method. It is important to point out that some other methods should be applied for systems with separable or difference kernels. Whereas, the AF transform can be used for solving systems of Volterra integro- differential equations with any kind of kernels. By comparing the results of other numerical methods such as homotopy perturbation method(HPM) [3], we conclude that the AF transform is more accurate, fast and reliable. Besides , AF transform does not require small parameters; thus, the limitations of the traditional perturbation methods can be eliminated, and the calculations are also simple and straight-forward. These advantages has been confirmed by employing two examples. Therefore, this method is a very effective tool for calculating the exact solu- tions of integro-differential equations systems.

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