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Research Article

Some Identities and Generating Functions for Padovan Numbers

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Abstract: In this paper, we introduce a symmetric endomorphism operator $\delta^{k}_{a_1a_2}$ allows us to obtain a new generating functions involving the product of Padovan numbers, k-Fibonacci numbers, k-Lucas numbers and Chebychev polynomials of first and second kind.

Keywords: Padovan numbers; k-Fibonacci numbers; k-Lucas numbers; Generating functions; Chebychev polynomial.

1. INTRODUCTION

Many authors have studied some special number by using homogeneous linear recurrence relations and their properties such as: Fibonacci numbers, Lucas numbers, Padovan numbers and Perrin numbers ; see for example, [1,2,3,4,5]. The Fibonacci numbers F_n is defined recurrently by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 1, F_1 = 1.$ It is well known that the ratio of two consecutive of Fibonacci numbers converges to $\alpha = \left(1+\sqrt{5}\right)/2$ [6]. In similar manner, the ratio of two consecutive Padovan numbers converges to Plastic numbers ρ [12]:

$$\rho = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$$

This value was firstly defined in 1924 by Gérard Cordonnier. The Padovan sequence is the sequence of integers P_n defined by the third-order recurrence relation $P_n = P_{n-2} + P_{n-3}, n \geq 3$ and the initial values: $P_0 = P_1 = P_2 = 1$ [7]. The first values of P_n are: 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37,49,…(sequence A000931 in [8]). The Padovan numbers are named after the mathematician Richard Padovan . There are many studies concerns about the Padovan numbers for example: In [9] Cereceda provied some determinantal representation of the Padovan numbers by using the Hessenberg matrices. In [10] Yilmaz and Taskara developed the matrix sequence that represent Padovan numbers and examined their properties. In this study, a new generating functions of the product of Padovan numbers, k-Fibonacci numbers and k-Lucas numbers are given by using some properties of symmetric functions.

2. SYMMETRIC FUNCTIONS

Definition 1:[11] Let k and n be two positive integer and $\{a_1, a_2, \dots, a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_2, \dots, a_n)$are defined by

$$e_k(a_1, a_2, \dots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k}^{\infty} a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}, \qquad 0 \leq k \leq n,$$

with $i_1, i_2, \dots, i_n = 0$ or 1.

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Definition 2:[11] Let k and n be two positive integer and $\{a_1, a_2, \dots, a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a_2, \dots, a_n)$ are defined by

$$h_k(a_1, a_2, \dots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k}^{\infty} a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}, \quad 0 \le k \le n,$$

with $i_1, i_2, \dots, i_n \ge 0$.

Remark 1: We set $e_0(a_1, a_2, \dots, a_n) = 1$ and $h_0(a_1, a_2, \dots, a_n) = 1$(by convention). For $k > n$ or $k < 0$, we set $e_k(a_1, a_2, \dots, a_n) = 0$ and $h_k(a_1, a_2, \dots, a_n)$=0.

Remark 2: Let $A = \{a_1, a_2, \dots, a_n\}$ an alphabet, we have

$$h_k^n(a_1, a_2, \dots, a_n) = h_k(a_1, a_2, \dots, a_n) = S_k(a_1 + a_2 + \cdots + a_n).$$

Definition 3:[12] Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\frac{\prod_{b\in B}(1 - bz)}{\prod_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - B)z^n, \tag{2.1}$$

with the condition $S_n(A - B)$=0 for $n < 0$ [13].

Remark 3: [14]Taking $A = \{0,0, \dots, 0\}$ in (2.1) gives

$$\sum_{n=0}^{\infty} S_n(-B)z^n = \prod_{b\in B}(1 - bz). \tag{2.2}$$

Definition 4:[15] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{a_i, a_{i+1}} f(a_1) = \frac{f(a_1, \dots, a_i, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_{i-1}, a_{i+1}, a_i, \dots, a_n)}{a_i - a_{i+1}}.$$

Definition 5:[16] The symmetrizing operator $\delta_{a_1 a_2}^k$ is defined by

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad for\ all\ k \in \mathbb{N}. \tag{2.3}$$

If $f(a_1) = a_1$, the operator (2.3) gives us

$$\delta_{a_1 a_2}^k f(a_1) = h_k(a_1, a_2) \text{ [17]}.$$

Proposition 1:[18.19] The relations

$$1) F_{k,-n} = (-1)^{n+1} F_{k,n},$$
$$2)\ L_{k,-n} = (-1)^n L_{k,n}.$$

For all $n \ge 0$.

3. On the Generating Functions of Some Padovan Numbers and Polynomails

The following Theorem is one of the key tools of the proof of our main results.

Theorem 1:[20] Let A and B be two alphabets, respectively, $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2\}$, then we have

$$\sum_{n=0}^{\infty} h_n^{(n)}(a_1, a_2, \dots, a_n) h_{k+n-1}^{(2)}(b_1, b_2) z^n$$
$$= \frac{\sum_{n=0}^{k-1} (-1)^n e_n(a_1, a_2, \dots, a_n) h_{k-n-1}^{(2)}(b_1, b_2) b_1^n b_2^n z^n}{\left(\sum_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \dots, a_n)(b_1 z)^n\right)\left(\sum_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \dots, a_n)(b_2 z)^n\right)}$$
$$- \frac{(b_1 b_2)^k \sum_{n=0}^{\infty} (-1)^{n+k+1} e_{n+k+1}(a_1, a_2, \dots, a_n) h_n^{(2)}(b_1, b_2) z^{n+1}}{\left(\sum_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \dots, a_n)(b_1 z)^n\right)\left(\sum_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \dots, a_n)(b_2 z)^n\right)}.$$

In this section, we use the previous theorem in order to derive a new generating function involving the product of Padovan numbers, k-Fibonacci numbers and Chebychev polynomial of first and second kind.

3.1. If $A = \{a_1, a_2, a_3\}, B = \{1.0\}$

Lemma 1. Given an alphabet $A = \{a_1, a_2, a_3\}$, we have

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)\, z^n = \frac{1}{\prod_{a\in A}(1 - az)}\,. \tag{3.1}$$

From (3.1) we can deduce the following formula :

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)\, z^n = \frac{z}{\prod_{a\in A}(1 - az)}, \tag{3.2}$$

with

$$\prod_{a\in A}(1 - az) = 1 - (a_1 + a_2 + a_3)z + (a_1a_2 + a_1a_3 + a_2a_3)z^2 - a_1a_2a_3z^3$$
$$= 1 - e_1(a_1, a_2, a_3)z + e_2(a_1, a_2, a_3)z^2 - e_3(a_1, a_2, a_3)z^3.$$

The substitution $\begin{cases} e_1(a_1, a_2, a_3) = 0 \\ e_2(a_1, a_2, a_3) = -1 \\ e_3(a_1, a_2, a_3) = 1 \end{cases}$ in (3.1) and (3.2) we obtain the following formulas, respectively:

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)\, z^n = \frac{1}{1 - z^2 - z^3}. \tag{3.3}$$

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)\, z^n = \frac{z}{1 - z^2 - z^3}. \tag{3.4}$$

By added the formula (3.3) to (3.4) we obtain

$$\sum_{n=0}^{\infty} \left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3)\right) z^n = \frac{1+z}{1-z^2-z^3},$$

which represents a generating function of Padovan numbers

Corollary 1: For $n \in \mathbb{N}$, we have

$$P_n = h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3).$$

3.2. If $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}$

Based on the theorem 1 ($k = 1{,}2$) we get the following Lemmas

Lemma 2:[21] Given two alphabets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$, we have

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)h_n^{(2)}(b_1, b_2)\, z^n = \frac{1 - b_1b_2e_2(a_1, a_2, a_3)z^2 + b_1b_2(b_1 + b_2)e_3(a_1, a_2, a_3)z^3}{\prod_{a\in A}(1 - ab_1z)\prod_{a\in A}(1 - ab_2z)}. \tag{3.5}$$

Lemma 3: Given two alphabets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$, we have

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)h_{n+1}^{(2)}(b_1, b_2)\, z^n = \frac{(b_1+b_2) - b_1b_2e_1(a_1, a_2, a_3)z + b_1^2b_2^2e_3(a_1, a_2, a_3)z^3}{\prod_{a\in A}(1 - ab_1z)\prod_{a\in A}(1 - ab_2z)}. \tag{3.6}$$

From (3.6) we can deduce

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)h_n^{(2)}(b_1, b_2)\, z^n = \frac{(b_1+b_2)z - b_1b_2e_1(a_1, a_2, a_3)z^2 + b_1^2b_2^2e_3(a_1, a_2, a_3)z^4}{\prod_{a\in A}(1 - ab_1z)\prod_{a\in A}(1 - ab_2z)}. \tag{3.7}$$

By replacing b_2 by $(-b_2)$in (3.5) and (3.7) we obtain, respectively, the following relationships

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)h_n^{(2)}(b_1, [-b_2])\, z^n = \frac{1 + b_1b_2e_2(a_1, a_2, a_3)z^2 - b_1b_2(b_1 - b_2)e_3(a_1, a_2, a_3)z^3}{\prod_{a\in A}(1 - ab_1z)\prod_{a\in A}(1 + ab_2z)}. \tag{3.8}$$

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1,a_2,a_3)h_n^{(2)}(b_1,[-b_2])\,z^n = \frac{(b_1-b_2)z + b_1b_2e_1(a_1,a_2,a_3)z^2 + b_1^2b_2^2e_3(a_1,a_2,a_3)z^4}{\prod_{a\in A}(1-ab_1z)\prod_{a\in A}(1+ab_2z)}, \quad (3.9)$$

with

$$\prod_{a\in A}(1-ab_1z)\prod_{a\in A}(1+ab_2z)$$

$$= 1-(b_1-b_2)e_1(a_1,a_2,a_3)z + \left(e_2(a_1,a_2,a_3)(b_1-b_2)^2 - b_1b_2\left(\left(e_1(a_1,a_2,a_3)\right)^2 - 2e_2(a_1,a_2,a_3)\right)\right)z^2$$

$$-\left(e_3(a_1,a_2,a_3)(b_1-b_2)^3 - b_1b_2(b_1-b_2)\left(e_2(a_1,a_2,a_3)e_1(a_1,a_2,a_3) - 3e_3(a_1,a_2,a_3)\right)\right)z^3 +$$

$$\left(-b_1b_2(b_1-b_2)^2e_3(a_1,a_2,a_3)e_1(a_1,a_2,a_3) + b_1^2b_2^2\left(\left(e_2(a_1,a_2,a_3)\right)^2 - 2e_3(a_1,a_2,a_3)e_1(a_1,a_2,a_3)\right)\right)z^4$$

$$-b_1^2b_2^2(b_1-b_2)e_3(a_1,a_2,a_3)e_2(a_1,a_2,a_3)z^5 - b_1^3b_2^3\left(e_3(a_1,a_2,a_3)\right)^2z^6.$$

The substitution $\begin{cases} e_1(a_1,a_2,a_3)=0 \\ e_2(a_1,a_2,a_3)=-1 \\ e_3(a_1,a_2,a_3)=1 \end{cases}$ and $\begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases}$ in (3.8)and (3.9) give

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1,a_2,a_3)h_n^{(2)}(b_1,[-b_2])\,z^n = \frac{1-z^2-kz^3}{1-(2+k^2)z^2-(3k+k^3)z^3+z^4+kz^5-z^6}. \quad (3.10)$$

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1,a_2,a_3)h_n^{(2)}(b_1,[-b_2])\,z^n = \frac{kz+z^4}{1-(2+k^2)z^2-(3k+k^3)z^3+z^4+kz^5-z^6}. \quad (3.11)$$

By added the formula (3.10) to (3.11) we obtain:

$$\sum_{n=0}^{\infty} h_n^{(2)}(b_1,[-b_2])\left(h_n^{(3)}(a_1,a_2,a_3)+h_{n-1}^{(3)}(a_1,a_2,a_3)\right)z^n = \frac{1+kz-z^2-kz^3+z^4}{1-(2+k^2)z^2-(3k+k^3)z^3+z^4+kz^5-z^6}, \quad (3.12)$$

which represents a new generating function, involving the product of Padovan numbers with k- Fibonacci numbers, such that: $P_nF_{k,n} = h_n^{(2)}(b_1,[-b_2])\left(h_n^{(3)}(a_1,a_2,a_3)+h_{n-1}^{(3)}(a_1,a_2,a_3)\right)$.

Proposition 2: For $n\in\mathbb{N}$, the new generating function of the product of Padovan numbers and k-Fibonacci numbers at negative indices is given by

$$\sum_{n=0}^{\infty} P_nF_{k,-n}\,z^n = \frac{-1+kz+z^2-kz^3-z^4}{1-(2+k^2)z^2+(3k+k^3)z^3+z^4-kz^5-z^6}. \quad (3.13)$$

Put $k=1$in (3.12)and (3.13) we deduce the following propositions

Proposition 3: For $n\in\mathbb{N}$, the new generating function of the product of Padovan numbers and Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} P_nF_n\,z^n = \frac{1+z-z^2-z^3+z^4}{1-3z^2-4z^3+z^4+z^5-z^6}.$$

Proposition 4: For $n\in\mathbb{N}$, the new generating function of the product of Padovan numbers and Fibonacci numbers at negative indices is given by

$$\sum_{n=0}^{\infty} P_nF_{-n}\,z^n = \frac{-1+z+z^2-z^3-z^4}{1-3z^2+4z^3+z^4-z^5-z^6}.$$

Theorem 2: For $n \in \mathbb{N}$, the new generating function of the product of k-Lucas numbers and Padovan numbers is given by

$$\sum_{n=0}^{\infty} P_n L_{k,n}\, z^n = \frac{2 + kz - (2 + k^2)z^2 - k(2 + k^2)z^3 + (2 + k^2)z^4}{1 - (2 + k^2)z^2 - (3k + k^3)z^3 + z^4 + kz^5 - z^6}. \quad (3.14)$$

Proof. We have

$$L_{k,n} = (2 + k^2)h_n^{(2)}(b_1, [-b_2]) - \mathrm{k}\, h_{n+1}^{(2)}(b_1, [-b_2])\ (see[22]),$$

then

$$\sum_{n=0}^{\infty} P_n L_{k,n}\, z^n = \sum_{n=0}^{\infty} \left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3)\right)\left((2 + k^2)h_n^{(2)}(b_1, [-b_2]) - \mathrm{k}\, h_{n+1}^{(2)}(b_1, [-b_2])\right) z^n$$

$$= (2 + k^2)\sum_{n=0}^{\infty} h_n^{(2)}(b_1, [-b_2])\left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3)\right) z^n -$$

$$k\sum_{n=0}^{\infty} h_{n+1}^{(2)}(b_1, [-b_2])\left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3)\right) z^n$$

$$= (2 + k^2)\sum_{n=0}^{\infty} P_n F_{k,n}\, z^n - k\sum_{n=0}^{\infty} h_{n+1}^{(2)}(b_1, [-b_2]) h_n^{(3)}(a_1, a_2, a_3)\, z^n$$

$$-k\sum_{n=0}^{\infty} h_{n+1}^{(2)}(b_1, [-b_2]) h_{n-1}^{(3)}(a_1, a_2, a_3)\, z^n$$

$$= (2 + k^2)\sum_{n=0}^{\infty} P_n F_{k,n}\, z^n - k\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)\frac{b_1^{n+2} - (-b_2)^{n+2}}{b_1 + b_2} z^n$$

$$-k\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)\frac{b_1^{n+2} - (-b_2)^{n+2}}{b_1 + b_2} z^n$$

$$= (2 + k^2)\sum_{n=0}^{\infty} P_n F_{k,n}\, z^n -$$

$$\frac{k}{b_1 + b_2}\left(b_1^2\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)(b_1 z)^n - b_2^2\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)(-b_2 z)^n\right)$$

$$-\frac{k}{b_1 + b_2}\left(b_1^2\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)(b_1 z)^n - b_2^2\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)(-b_2 z)^n\right)$$

$$= \frac{(2 + k^2)(1 + kz - z^2 - kz^3 + z^4)}{1 - (2 + k^2)z^2 - (3k + k^3)z^3 + z^4 + kz^5 - z^6} - \frac{k^2 + k(k^2 + 1)z}{1 - (2 + k^2)z^2 - (3k + k^3)z^3 + z^4 + kz^5 - z^6}$$

$$= \frac{2+kz-(2+k^2)z^2-k(2+k^2)z^3+(2+k^2)z^4}{1-(2+k^2)z^2-(3k+k^3)z^3+z^4+kz^5-z^6}.$$

This completes the proof.

Proposition 5: *For* $n \in \mathbb{N}$, the new generating function of the product of Padovan numbers and k-Lucas numbers at negative indices is given by

$$\sum_{n=0}^{\infty} P_n L_{k,-n}\, z^n = \frac{2 - kz - (2 + k^2)z^2 + k(2 + k^2)z^3 + (2 + k^2)z^4}{1 - (2 + k^2)z^2 + (3k + k^3)z^3 + z^4 - kz^5 - z^6}.$$

Put $k = 1$ in (3.14) we deduce the following proposition

Proposition 6: *For* $n \in \mathbb{N}$, the new generating function of the product of Lucas numbers and Padovan numbers is given by

$$\sum_{n=0}^{\infty} P_n L_n\, z^n = \frac{2+z-3z^2-3z^3+3z^4}{1-3z^2-4z^3+z^4+z^5-z^6}.$$

Replacing b_1 by $2b_1$ and b_2 by ($-2b_2$) in (3.5) and (3.7) we obtain

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1,a_2,a_3)h_n^{(2)}(2b_1,[-2b_2])\, z^n = \frac{1+4b_1b_2e_2(a_1,a_2,a_3)z^2 - 8b_1b_2(b_1-b_2)e_3(a_1,a_2,a_3)z^3}{\prod_{a\in A}(1-2ab_1z)\prod_{a\in A}(1+2ab_2z)} . \quad (3.15)$$

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1,a_2,a_3)h_n^{(2)}(2b_1,[-2b_2])\, z^n = \frac{2(b_1-b_2)z+4b_1b_2e_1(a_1,a_2,a_3)z^2+16b_1^2b_2^2e_3(a_1,a_2,a_3)z^4}{\prod_{a\in A}(1-2ab_1z)\prod_{a\in A}(1+2ab_2z)} . \quad (3.16)$$

With

$$\prod_{a\in A}(1-2ab_1z)\prod_{a\in A}(1+2ab_2z)$$

$$= 1-2(b_1-b_2)e_1(a_1,a_2,a_3)z+\left(4e_2(a_1,a_2,a_3)(b_1-b_2)^2-4b_1b_2\left(\left(e_1(a_1,a_2,a_3)\right)^2-2e_2(a_1,a_2,a_3)\right)\right)z^2$$

$$-\left(8e_3(a_1,a_2,a_3)(b_1-b_2)^3-8b_1b_2(b_1-b_2)\left(e_2(a_1,a_2,a_3)e_1(a_1,a_2,a_3)-3e_3(a_1,a_2,a_3)\right)\right)z^3+$$

$$\left(-16b_1b_2(b_1-b_2)^2e_3(a_1,a_2,a_3)e_1(a_1,a_2,a_3)+16b_1^2b_2^2\left(\left(e_2(a_1,a_2,a_3)\right)^2-2e_3(a_1,a_2,a_3)e_1(a_1,a_2,a_3)\right)\right)z^4$$

$$-32b_1^2b_2^2(b_1-b_2)e_3(a_1,a_2,a_3)e_2(a_1,a_2,a_3)z^5-64b_1^3b_2^3\left(e_3(a_1,a_2,a_3)\right)^2z^6.$$

By making the following restrictions $\begin{cases} e_1(a_1,a_2,a_3)=0 \\ e_2(a_1,a_2,a_3)=-1 \\ e_3(a_1,a_2,a_3)=1 \end{cases}$ and $\begin{cases} b_1-b_2=x \\ 4b_1b_2=-1 \end{cases}$ in (3.15)and (3.16) we get a new generating functions

$$\sum_{n=0}^{\infty} h_n^{(3)}(a_1,a_2,a_3)h_n^{(2)}(2b_1,[-2b_2])\, z^n = \frac{1+z^2+2xz^3}{1+(-4x^2+2)z^2-(8x^3-6x)z^3+z^4+2xz^5+z^6} . \quad (3.17)$$

$$\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1,a_2,a_3)h_n^{(2)}(2b_1,[-2b_2])\, z^n = \frac{2xz+z^4}{1+(-4x^2+2)z^2-(8x^3-6x)z^3+z^4+2xz^5+z^6} . \quad (3.18)$$

By added the formula (3.17) to (3.18) we obtain

$$\sum_{n=0}^{\infty} h_n^{(2)}(2b_1,[-2b_2])\left(h_n^{(3)}(a_1,a_2,a_3)+h_{n-1}^{(3)}(a_1,a_2,a_3)\right)z^n$$
$$= \frac{1+2xz+z^2+2xz^3+z^4}{1+(-4x^2+2)z^2-(8x^3-6x)z^3+z^4+2xz^5+z^6},$$

which represents a new generating function of the product of Padovan numbers and Chebychev polynomial of the second kind such that : $P_nU_n = h_n^{(2)}(2b_1,[-2b_2])\left(h_n^{(3)}(a_1,a_2,a_3)+h_{n-1}^{(3)}(a_1,a_2,a_3)\right)$.

Theorem 3: *For* $n\in\mathbb{N}$, the new generating function of the product of Padovan numbers and Chebychev polynomial of the first kind is given by

$$\sum_{n=0}^{\infty} P_nT_n(x)\, z^n = \frac{1+xz+(1-2x^2)z^2+2x(1-2x^2)z^3+(1-2x^2)z^4}{1-(4x^2-2)z^2-(8x^3-6x)z^3+z^4+2xz^5+z^6}.$$

Proof. We have

$$T_n(x) = h_n^{(2)}(2b_1,[-2b_2]) - xh_{n-1}^{(2)}(2b_1,[-2b_2]) \quad (see\ [23]),$$

then

$$\sum_{n=0}^{\infty} P_nT_n\, z^n = \sum_{n=0}^{\infty}\left(h_n^{(3)}(a_1,a_2,a_3)+h_{n-1}^{(3)}(a_1,a_2,a_3)\right)\left(h_n^{(2)}(2b_1,[-2b_2])-xh_{n-1}^{(2)}(2b_1,[-2b_2])\right)z^n$$

$$= \sum_{n=0}^{\infty} h_n^{(2)}(2b_1, [-2b_2]) \left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3) \right) z^n - x \sum_{n=0}^{\infty} h_{n-1}^{(2)}(2b_1, [-2b_2]) \left(h_n^{(3)}(a_1, a_2, a_3) + h_{n-1}^{(3)}(a_1, a_2, a_3) \right) z^n$$

$$= \sum_{n=0}^{\infty} P_n U_n(x)\, z^n - x \sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3) h_{n-1}^{(2)}(2b_1, [-2b_2])\, z^n - x \sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3) h_{n-1}^{(2)}(2b_1, [-2b_2])\, z^n$$

$$= \sum_{n=0}^{\infty} P_n U_n(x)\, z^n - \frac{x}{2b_1 + 2b_2} \left(\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)((2b_1)^n - (-2b_2)^n)\, z^n \right) - \frac{x}{2b_1 + 2b_2} \left(\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)((2b_1)^n - (-2b_2)^n)\, z^n \right)$$

$$= \sum_{n=0}^{\infty} P_n U_n(x)\, z^n - \frac{x}{2b_1 + 2b_2} \left(\sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)(2b_1 z)^n - \sum_{n=0}^{\infty} h_n^{(3)}(a_1, a_2, a_3)(-2b_2 z)^n \right) - \frac{x}{2b_1 + 2b_2} \left(\sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)(2b_1 z)^n - \sum_{n=0}^{\infty} h_{n-1}^{(3)}(a_1, a_2, a_3)(-2b_2 z)^n \right)$$

$$= \frac{1 + 2xz + z^2 + 2xz^3 + z^4}{1 - (4x^2 - 2)z^2 - (8x^3 - 6x)z^3 + z^4 + 2xz^5 + z^6} - \frac{xz + 2x^2 z^2 + 4x^3 z^3 + 2x^2 z^4}{1 - (4x^2 - 2)z^2 - (8x^3 - 6x)z^3 + z^4 + 2xz^5 + z^6}$$

$$= \frac{1 + xz + (1 - 2x^2)z^2 + 2x(1 - 2x^2)z^3 + (1 - 2x^2)z^4}{1 - (4x^2 - 2)z^2 - (8x^3 - 6x)z^3 + z^4 + 2xz^5 + z^6}.$$

This completes the proof.

CONCLUSION

In this paper, by making use of Theorem 1 we have written some new generating functions for the products of Padovan numbers, k-Fibonacci numbers, k-Lucas numbers and Chebychev polynomials of first and second kinds.

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REFERENCES

[1] A. G. Shannon, P. G. Anderson and A. F. Horadam, Properties of Cordonnier, Perrin and Van der Laan numbers, International Journal of Mathematical Education in Science and Technology 37(7) (2006), 825-831.

[2] A. G. Shannon. A. F. Horadam, Generating functions for powers of third order recurrence sequences, Duke Math. J. 38 (1971), 791-794.

[3] S. Falcon and A. Plaza, On the Fibonacci k-numbers, Chaos, Solitons & fractal. 32 (2007), 1615-1624.

[4] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley and Sons Inc., NY, 2001.

[5] Y. Yazlik and N.Taskara, A note on generalized k-Horadam sequence, Comput. Math. Appl. 63(2012), 36-41.

[6] N.Yilmaz and N. Taskara, On the Negatively Subscripted Padovan and Perrin Matrix Sequences, Communications in Mathematic and Applications. 5 (2) (2014), 59-72.

[7] I. Stewart, Tales of a neglected number, Sci. Am.274 (1996), 102-103.

[8] N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electornically at https:// oeis.org.

[9] J. L. Cereceda, Determinantal represantations for generalized Fibonacci and tribonacci numbers, Int. J. Contemp. Math. Sci. 9(6) (2014), 269-285.

[10] F.Yilmaz and N. taskara, Matrix sequences in terms of Padovan and Perrin numbers J. Appl. Math (2013), Article ID941673.
[11] A. Boussayoud and N. Harrouche , Complete Symmetric Functions and k-Fibonacci Numbers, Commun. appl.Anal. 20 (2016),457-465.
[12] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle, Adv. math. 103 (1994), 180-195.
[13] A. Boussayoud, M. Chelgham, S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomails, Turkish Journal of Analysis and Number Theory. 6(3) (2018), 93-97.
[14] A. Boussayoud, S. Boughaba, M. Kerada, Generating Functions k-Fibonacci and k-Jacobsthal numbers at negative indices, Electron. J. Math. Analysis Appl. 6(2) (2018), 195-202.
[15] A. Boussayoud, A. Abderrezzak and M, Kerada, Some applications of symmetric functions, Integers. 15A#48 (2015), 1-7.
[16] A. Boussayoud , M. Kerada and R. Sahali, Symmetrizing Operations on Some Orthogonal Polynomials , Int. Electron. J. Pure Appl. Math.9 (2015), 191-199.
[17] A. Boussaoud, On some identities and generating functions for Pell-Lucas numbers, Online.J.Anal. Comb. 12#1 (2017), 1-10.
[18] S. Falcon, On the k-Lucas Numbers of Arithmetic Indexes, Appl. Math. 3 (2012), 1202-1206.
[19] S.Falcon, On the complex k-Fibonacci numbers, Cogent Math. 3 (2016), 1-9.
[20] A. abderrezzak, M. Kerada and A. Boussayoud, Generalization of some Hadamard product, Communications in Applied Analysis , 20 (2016), 301-306.
[21] A. Boussayoud, A. A. Abderrezzak, Complet Homogeneous Symmetric Functions and Hadamard Product, Ars Comb. 144 (2019), 81-90.
[22] A. Boussayoud, S. Boughaba, On Some Identities and Generating Functions for k-Pell sequences and Chebyshev polynomials, Online J. Anal. Comb. 14#3 (2019), 1-13.
[23] A . Boussayoud, S. Boughaba, M. Kerada, S. Araci and M. Acikgoz, Generating function of binary products of k-Fibonacci and orthogonal polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 113 (2019), 2575-2586.