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Research Article

A Note on the Symmetric Relations of *q*-Genocchi Polynomials under Symmetric Group of Degree *n*

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Abstract: In the paper, we deal mainly with some new relations of *q*-Genocchi polynomials. From those relations, we deduce that these relations can be stated by symmetric group of degree n, denoted by S_n.

Keywords: Symmetric identities; *q*-Genocchi polynomials; Fermionic *p*-adic *q*-integral on $\mathbb{Z}_p$; Symmetric group of degree S_n.

INTRODUCTION

Throughout of the paper, we make use of the following notations:

$$\mathbb{N} := \{1,2,3,\cdots\} \text{ and } \mathbb{N}^* := \mathbb{N} \cup \{0\}.$$

Here $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of *p*-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of *p*-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, where p is a fixed odd prime number. For d an odd positive number with $(p,d)=1$, set

$$X := X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^n\mathbb{Z},\ X_1 = \mathbb{Z}_p$$

and

$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \left(mod\ dp^n\right)\}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^n$. See, for details, $[1-18]$.

The *p*-adic normalized absolute value is given by $|p|_p = p^{-1}$. Note that "q" can be considered as an indeterminate a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a *p*-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

We now give the definition of *q*-number, as follows:

$$[x]_q = \begin{cases} \dfrac{q^x - 1}{q-1}, & \text{if } q \ne 1 \\ x, & \text{if } q = 1. \end{cases}$$

See $[1-18, 20]$.

The *q*-Volkenborn integral (or sometimes called *p*-adic *q*-integral on $\mathbb{Z}_p$) of a function $f \in UD(\mathbb{Z}_p)$ is originally defined by Kim $[16]$, as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{n\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^n-1} f(x) q^x. \tag{1.1}$$

The fermionic *p*-adic *q*-deformed integral on $\mathbb{Z}_p$ is also defined by Kim $[10,15]$, as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \frac{[2]_q}{2} \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x) (-1)^x q^x. \tag{1.2}$$

Because of the Eq. (1.1) and the Eq.(1.2), it can be written symbolically as

$$\lim_{q\to -q} I_q(f) = I_{-q}(f).$$

By the Eq. (1.2), we have the following equality which plays extremely important role in order to get the new generalizations of some special polynomials:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q\, f(0)$$

where $f_1(x) = f(x+1)$. See $[1-20]$ for a systematic works about these topics.

In [4], Araci and Acikgoz gave the generating function of *q*-Genocchi polynomials $G_{n,q}(x)$, with respect to μ_{-q}, as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} &= t \int_{\mathbb{Z}_p} q^{-y} e^{t[y+x]_q}\, d\mu_{-q}(y) \qquad (1.3) \\ &= [2]_q\, t \sum_{m=0}^{\infty} (-1)^m\, e^{t[m+x]_q}. \end{aligned}$$

If we take $x = 0$ into the Eq. (1.3), it becomes $G_{n,q}(0) := G_{n,q}$ that are called n-th *q*-Genocchi number. As q approaches to 1 in the Eq. (1.3) gives

$$\lim_{q\to 1} G_{n,q}(x) := G_n(x)$$

where $G_n(x)$ is known as familiar Genocchi polynomials cf. [3],[4],[5],[7],[20].

Recently, Kim *et al.*[13] have introduced a beautiful method in order to construct symmetric identities of some well known special polynomials under symmetric groups. The symmetric identities of *q*-Euler polynomials and *q*-Bernoulli polynomials under symmetric group of degree n was also given by Dolgy *et al*.[9] and Kim *et al*.[14]. Moreover, Duran *et al*.[7] obtained symmetric identities of q-Genocchi polynomials under symmetric group of degree four. Now also, we extend symmetric identities of q-Genocchi polynomials under symmetric group of degree four to symmetric identities of q-Genocchi polynomials under symmetric group of degree n using the useful method of Kim *et al*.[14] which we state in the next section.

ON THE SYMMETRIC RELATIONS OF *q*-GENOCCHI POLYNOMIALS UNDER S_n

Let $w_i \in \mathbb{N}$ with $w_i \equiv 1 (mod\, 2)$ for $i \in \{1, 2, \cdots, n\}$. Then, by the Eqs. (1.2) and (1.3), we have

$$\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1} w_j\right)} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}}(y) \tag{2.1}$$

$$= \frac{[2]_q}{2}\lim_{n\to\infty}\sum_{y=0}^{p^n-1}(-1)^{y\left(\prod_{j=1}^{n-1} w_j\right)} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}$$

$$= \frac{[2]_q}{2}\lim_{n\to\infty}\sum_{m=0}^{w_4-1}\sum_{y=0}^{p^n-1}(-1)^{m+y} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.$$

Applying

$$\frac{2}{[2]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{l=1}^{n-1} w_l\right)}$$

to the both sides of the Eq. (2.1), we derive that

$$I = \frac{2}{[2]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{l=1}^{n-1} w_l\right)} \tag{2.2}$$

$$\times\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1} w_j\right)} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}}(y)$$

$$= \lim_{n\to\infty}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_4-1}\sum_{y=0}^{p^n-1}(-1)^{\sum_{l=1}^{n-1} w_l+m+y}$$

$$\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n-1} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.$$

We see that the Eq. (2.2) is invariant under any permutation $\sigma \in S_n$. Therefore, we state as follows:

$$\frac{2}{[2]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}w_{\sigma(s)}\right)}$$
$$\times\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} e^{\left[\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n}w_{\sigma(j)}\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t} d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(y)$$

in which σ lies in S_n. From this, we have the following theorem.

Theorem 1 *Let* $w_i \in \mathbb{N}$ *with* $w_i \equiv 1(mod\, 2)$ *for* $i \in \{1,2,\cdots,n\}$. *Then the following*

$$\frac{2}{[2]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}w_{\sigma(s)}\right)}$$
$$\times\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} e^{\left[\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n}w_{\sigma(j)}\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t} d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(y)$$

holds true for any $\sigma \in S_n$.

From the definition of q-number $[x]_q$, we have

$$\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q \tag{2.3}$$
$$=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+\frac{w_n}{w_1}k_1+\ldots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1w_2\cdots w_{n-1}}}$$
$$=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}.$$

By (2.3), we obtain that

$$\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1}w_j\right)} e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q t} d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y) \tag{2.4}$$
$$=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1}w_j\right]_q^m\left(\int_{\mathbb{Z}_p} q^{-y\left(\prod_{j=1}^{n-1}w_j\right)}\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)\right)\frac{t^m}{m!}$$
$$=\frac{1}{m+1}\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1}w_j\right]_q^m G_{m+1,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right)\frac{t^m}{m!}.$$

It directly follows from the Eq. (2.4) that

$$\int_{Z_p} q^{-y\left(\prod_{j=1}^{n-1} w_j\right)} \left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j \right]_q^m d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}}(y) \tag{2.5}$$

$$= \frac{\left[\prod_{j=1}^{n-1} w_j\right]_q^m}{m+1} G_{m+1, q^{w_1 w_2 \cdots w_{n-1}}} \left(w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \quad (m \geq 0).$$

Hence, by Theorem 1 and (2.5), we get the following theorem.

Theorem 2 *Let* $w_i \in \mathbb{N}$ *with* $w_i \equiv 1 (mod\, 2)$ *for* $i \in \{1, 2, \cdots, n\}$. *Then the following*

$$\frac{2}{[2]_q} \frac{\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^m}{m+1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} (-1)^{\left(\sum_{s=1}^{n-1} w_{\sigma(s)}\right)} G_{m+1, q^{w_1 w_2 \cdots w_{n-1}}} \left(w_{\sigma(n)} x + \sum_{j=1}^{n-1} \frac{w_{\sigma(n)}}{w_{\sigma(j)}} k_j \right)$$

holds true for any $\sigma \in S_n$ and $m \geq 0$.

It follows from the definition of $[x]_q$ and binomial theorem that

$$\left[y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right]_{q^{w_1 w_2 \cdots w_{n-1}}}^m \tag{2.6}$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q} \right)^{m-l} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{m-l}$$

$$\times q^{l w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i \right) k_j} [y + w_n x]_{q^{w_1 w_2 \cdots w_{n-1}}}^l .$$

Applying $\int_{Z_p} q^{-y\left(\prod_{j=1}^{n-1} w_j\right)} d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}}(y)$ to the both sides of the above gives

$$\int_{Z_p} q^{-y\left(\prod_{j=1}^{n-1} w_j\right)} \left[y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right]_{q^{w_1 w_2 \cdots w_{n-1}}}^m d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}}(y) \tag{2.7}$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q} \right)^{m-l} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{m-l}$$

$$\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\int_{\mathbb{Z}_p}q^{-y\left(\prod_{j=1}^{n-1}w_j\right)}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}^{l}\,d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)$$

$$=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$

$$\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\frac{G_{l+1,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right)}{l+1}.$$

By the Eq. (2.7), we have

$$\frac{2}{[2]_q}\left[\prod_{j=1}^{n-1}w_j\right]_q^m\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{l=1}^{n-1}w_l\right)} \tag{2.8}$$

$$\times\int_{\mathbb{Z}_p}q^{-y\left(\prod_{j=1}^{n-1}w_j\right)}\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)$$

$$=\sum_{l=0}^{m}\binom{m}{l}\frac{2}{[2]_q}\left[\prod_{j=1}^{n-1}w_j\right]_q^{l}[w_n]_q^{m-l}\frac{G_{l+1,q^{w_1w_2\cdots w_{n-1}}}(w_nx)}{l+1}$$

$$\times\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{l=1}^{n-1}w_l\right)}q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$

$$=\sum_{l=0}^{m}\binom{m}{l}\frac{2}{[2]_q}\left[\prod_{j=1}^{n-1}w_j\right]_q^{l}[w_n]_q^{m-l}\frac{G_{l+1,q^{w_1w_2\cdots w_{n-1}}}(w_nx)}{l+1}U_{m,q^{w_n}}(w_1,w_2,\ldots,w_{n-1}\,|\,l),$$

where

$$U_{m,q^{w_n}}(w_1,w_2,\ldots,w_{n-1}\,|\,l)$$

$$=\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{l=1}^{n-1}w_l\right)}q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}.$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 3 *Let* $w_i\in\mathbb{N}$ *with* $w_i\equiv 1(mod\,2)$ *for* $i\in\{1,2,\cdots,n\}$ *and let* $m\geq 0$. *Then the following expression*

$$\frac{2}{[2]_q}\sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q^{l}\left[w_{\sigma(n)}\right]_q^{m-l}\frac{G_{l+1,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\ldots\sigma(n-1)}}}(w_{\sigma(n)}x)}{l+1}U_{m,q^{w_{\sigma(n)}}}(w_{\sigma(1)},w_{\sigma(2)},\ldots,w_{\sigma(n-1)}\,|\,l)$$

holds true for some $\sigma\in S_n$.

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