Analytical study of the Difference Equation
$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(\pm 1 + x_n x_{n-6})}$$

Abdualrazaq Sanbo [1,2,*], Elsayed M. Elsayed [2,3] and Faris Alzahrani [2]

[1] General studies department, Jeddah College of Telecom and Electronics, TVTC, B.P. 2816, Jeddah 21461, Saudi Arabia
[2] Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract: This paper is devoted to find the form of the solutions of the following rational difference equations:
$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(\pm 1 + x_n x_{n-6})}, \quad n = 0,1,\ldots,$$
where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers. Also, we give specific form of the solutions of two cases of this equation.

1 Introduction

Our aim in this paper is to investigate the behavior of the solution of the following nonlinear difference equation
$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(\pm 1 + x_n x_{n-6})}, \quad n = 0,1,\ldots, \tag{1}$$
where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers.

The study and solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding. Recently, there has been a lot of interest in studying the qualitative properties of rational recursive sequences, Furthermore diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory. However, there have not been any suitable general method to deal with the global behavior of rational difference equations of high order so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Many researchers have investigated the behavior of the solution of difference equations, for example,

Elsayed et al. [23] has obtained results concerning the dynamics and solution of the rational difference equation
$$x_{n+1} = \frac{x_{n-1} x_{n-5}}{x_{n-3}(\pm 1 \pm x_{n-1} x_{n-5})}.$$

Elsayed et al. [27] has obtained results concerning The dynamics and the solutions of the rational difference equation
$$x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}(\pm 1 \pm x_n x_{n-4})}.$$

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* Correspondence: sanbo@cte.edu.sa

Aloqeili [3] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Simsek et al. [37] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

Çinar [7,8,9] got the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$$

In [28], Ibrahim got the form of the solution of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$$

Karatas et al. [30] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}$$

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f: I^{k+1} \rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0,1,\ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\bar{x} \in I$ is called an equilibrium point of Eq. (2) if $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq. (2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2. (Stability)

• The equilibrium point $\bar{x}$ of Eq. (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$$

we have $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$.

• The equilibrium point $\bar{x}$ of Eq. (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$$

we have $\lim_{n \to \infty} x_n = \bar{x}$.

• The equilibrium point $\bar{x}$ of Eq. (2) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

• The equilibrium point $\bar{x}$ of Eq. (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (2).

• The equilibrium point $\bar{x}$ of Eq. (2) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Eq. (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} y_{n-i}$$

Theorem 1. Assume that $p, q \in \mathbb{R}$ and $k \in \{0,1,2,\ldots\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, n = 0,1,\ldots.$$

Remark 1. The theorem can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0,1,\ldots, \tag{3}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{0,1,2,\ldots\}$. Then Eq. (3) is asymptotically stable provided that $\sum_{i=0}^{k} |p_i| < 1$.

Our goal in this section is to find a specific form of the solutions of some special cases of Eq. (1) when , a, b, c and d are integers and give numerical examples of each case.

2 First Case: On the Difference Equation $x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(1+x_n x_{n-6})}$

In this subsection we study the following special case of Eq. (1):

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(1+x_n x_{n-6})}, \quad n = 0,1,\ldots, \tag{4}$$

where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzeros real numbers.

Theorem 2. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (4). Then for $n = 0,1,\ldots$,

$$x_{12n-6} = x_{-6} \prod_{i=0}^{n-1} \left(\frac{1+(12i)x_{-6}x_0}{1+(12i+6)x_{-6}x_0}\right), \qquad x_{12n-5} = x_{-5} \prod_{i=0}^{n-1} \left(\frac{1+(12i+1)x_{-6}x_0}{1+(12i+7)x_{-6}x_0}\right),$$

$$x_{12n-4} = x_{-4} \prod_{i=0}^{n-1} \left(\frac{1+(12i+2)x_{-6}x_0}{1+(12i+8)x_{-6}x_0}\right), \qquad x_{12n-3} = x_{-3} \prod_{i=0}^{n-1} \left(\frac{1+(12i+3)x_{-6}x_0}{1+(12i+9)x_{-6}x_0}\right),$$

$$x_{12n-2} = x_{-2} \prod_{i=0}^{n-1} \left(\frac{1+(12i+4)x_{-6}x_0}{1+(12i+10)x_{-6}x_0}\right), \qquad x_{12n-1} = x_{-1} \prod_{i=0}^{n-1} \left(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+11)x_{-6}x_0}\right),$$

$$x_{12n} = x_0 \prod_{i=0}^{n-1} \left(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right), \qquad x_{12n+1} = \frac{x_{-6}x_0}{x_{-5}(1+x_{-6}x_0)} \prod_{i=0}^{n-1} \left(\frac{1+(12i+7)x_{-6}x_0}{1+(12i+13)x_{-6}x_0}\right),$$

$$x_{12n+2} = \frac{x_{-6}x_0}{x_{-4}(1+2x_{-6}x_0)} \prod_{i=0}^{n-1} \left(\frac{1+(12i+8)x_{-6}x_0}{1+(12i+14)x_{-6}x_0}\right), \quad x_{12n+3} = \frac{x_{-6}x_0}{x_{-3}(1+3x_{-6}x_0)} \prod_{i=0}^{n-1} \left(\frac{1+(12i+9)x_{-6}x_0}{1+(12i+15)x_{-6}x_0}\right),$$

$$x_{12n+4} = \frac{x_{-6}x_0}{x_{-2}(1+4x_{-6}x_0)} \prod_{i=0}^{n-1} \left(\frac{1+(12i+10)x_{-6}x_0}{1+(12i+16)x_{-6}x_0}\right), \quad x_{12n+5} = \frac{x_{-6}x_0}{x_{-1}(1+5x_{-6}x_0)} \prod_{i=0}^{n-1} \left(\frac{1+(12i+11)x_{-6}x_0}{1+(12i+17)x_{-6}x_0}\right).$$

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$x_{12n-18} = x_{-6} \prod_{i=0}^{n-2} \left(\frac{1+(12i)x_{-6}x_0}{1+(12i+6)x_{-6}x_0}\right), \qquad x_{12n-17} = x_{-5} \prod_{i=0}^{n-2} \left(\frac{1+(12i+1)x_{-6}x_0}{1+(12i+7)x_{-6}x_0}\right),$$

$$x_{12n-16} = x_{-4} \prod_{i=0}^{n-2} \left(\frac{1+(12i+2)x_{-6}x_0}{1+(12i+8)x_{-6}x_0}\right), \qquad x_{12n-15} = x_{-3} \prod_{i=0}^{n-2} \left(\frac{1+(12i+3)x_{-6}x_0}{1+(12i+9)x_{-6}x_0}\right),$$

$$x_{12n-14} = x_{-2} \prod_{i=0}^{n-2} \left(\frac{1+(12i+4)x_{-6}x_0}{1+(12i+10)x_{-6}x_0}\right), \qquad x_{12n-13} = x_{-1} \prod_{i=0}^{n-2} \left(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+11)x_{-6}x_0}\right),$$

$$x_{12n-12} = x_0 \prod_{i=0}^{n-2} \left(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right), \qquad x_{12n-11} = \frac{x_{-6}x_0}{x_{-5}(1+x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+7)x_{-6}x_0}{1+(12i+13)x_{-6}x_0}\right),$$

$$x_{12-10} = \frac{x_{-6}x_0}{x_{-4}(1+2x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+8)x_{-6}x_0}{1+(12i+14)x_{-6}x_0}\right), \quad x_{12n-9} = \frac{x_{-6}x_0}{x_{-3}(1+3x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+9)x_{-6}x_0}{1+(12i+15)x_{-6}x_0}\right),$$

$$x_{12n-8} = \frac{x_{-6}x_0}{x_{-2}(1+4x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+10)x_{-6}x_0}{1+(12i+16)x_{-6}x_0}\right), \quad x_{12n-7} = \frac{x_{-6}x_0}{x_{-1}(1+5x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+11)x_{-6}x_0}{1+(12i+17)x_{-6}x_0}\right).$$

Now, using the main Eq. (4), one has

$$x_{12n-6} = \frac{x_{12n-7}x_{12n-13}}{x_{12n-12}(1+x_{12n-7}x_{12n-13})}$$

$$= \frac{\frac{x_{-6}x_0}{x_{-1}(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1+(12i+11)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})x_{-1}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+11)x_{-6}x_0})}{x_0\prod_{i=0}^{n-2}(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0})\left(1+\frac{x_{-6}x_0}{x_{-1}(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}\frac{1+(12i+11)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})x_{-1}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+11)x_{-6}x_0})\right)}$$

$$= \frac{\frac{x_{-6}}{(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})}{\prod_{i=0}^{n-2}(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0})\left(1+\frac{x_{-6}x_0}{(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1+(12i+12)x_{-6}x_0}{1+(12i+6)x_{-6}x_0})\frac{\frac{x_{-6}}{(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})}{\left(1+\frac{x_{-6}x_0}{(1+5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1+(12i+5)x_{-6}x_0}{1+(12i+17)x_{-6}x_0})\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1+(12i+12)x_{-6}x_0}{1+(12i+6)x_{-6}x_0})\frac{x_{-6}\frac{1}{1+(12n-7)x_{-6}x_0}}{\left(1+\frac{x_{-6}x_0}{(1+(12n-7)x_{-6}x_0)}\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1+(12i+12)x_{-6}x_0}{1+(12i+6)x_{-6}x_0})\frac{x_{-6}}{1+(12n-6)x_{-6}x_0}.$$

Hence, we have

$$x_{12n-6} = x_{-6}\prod_{i=0}^{n-1}(\frac{1+(12i)x_{-6}x_0}{1+(12i+6)x_{-6}x_0}).$$

Similarly, using the main Eq. (4), one has

$$x_{12n-5} = \frac{x_{12n-6} x_{12n-12}}{x_{12n-11}(1+x_{12n-6} x_{12n-12})}$$

$$= \frac{x_{-6} \prod_{i=0}^{n-1} \left(\frac{1+(12i)x_{-6}x_0}{1+(12i+6)x_{-6}x_0}\right) x_0 \prod_{i=0}^{n-2} \left(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right)}{\frac{x_{-6}x_0}{x_{-5}(1+x_{-6}x_0)} \prod_{i=0}^{n-2} \left(\frac{1+(12i+7)x_{-6}x_0}{1+(12i+13)x_{-6}x_0}\right) \left(1+x_{-6} \prod_{i=0}^{n-1} \left(\frac{1+(12i)x_{-6}x_0}{1+(12i+6)x_{-6}x_0}\right) x_0 \prod_{i=0}^{n-2} \left(\frac{1+(12i+6)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right)\right)}$$

$$= x_{-5}(1+x_{-6}x_0) \prod_{i=0}^{n-2} \left(\frac{1+(12i+13)x_{-6}x_0}{1+(12i+7)x_{-6}x_0}\right) \frac{\left(\frac{1+(12n-12)x_{-6}x_0}{1+(12n-6)x_{-6}x_0}\right) \prod_{i=0}^{n-2}\left(\frac{1+(12i)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right)}{\left(1+x_{-6}x_0\left(\frac{1+(12n-12)x_{-6}x_0}{1+(12n-6)x_{-6}x_0}\right)\prod_{i=0}^{n-2}\left(\frac{1+(12i)x_{-6}x_0}{1+(12i+12)x_{-6}x_0}\right)\right)}$$

$$= x_{-5} \prod_{i=0}^{n-1} \left(\frac{1+(12i+1)x_{-6}x_0}{1+(12i+7)x_{-6}x_0}\right) \frac{1+x_{-6}x_0}{1+(12n-5)x_{-6}x_0}.$$

Hence, we have

$$x_{12n-5} = x_{-5} \prod_{i=0}^{n-1} \left(\frac{1+(12i+1)x_{-6}x_0}{1+(12i+7)x_{-6}x_0}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 3. Eq. (4) has one equilibrium point $\bar{x} = 0$ and this equilibrium point is not locally asymptotically stable.

Proof. In this section we investigate the local stability character of the solutions of Eq. (4). Equation (4) has a unique positive equilibrium point and is given by

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}(1+\bar{x}^2)} = \frac{\bar{x}}{1+\bar{x}^2},$$

or also

$$1 = 1 + \bar{x}^2,$$

then the unique equilibrium point is given by $\bar{x} = 0$.

Define the following function

$$f: (0,\infty)^3 \to (0,\infty)$$
$$f(u,v,w) = \frac{uw}{v(1+uw)}.$$

Therefore it follows that

$$f_u(u,v,w) = \frac{w}{v(1+uw)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2(1+uw)}, \quad f_w(u,v,w) = \frac{u}{v(1+uw)^2}.$$

Then

$$f_u(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1+\bar{x}^2)^2} = 1, \quad f_v(\bar{x},\bar{x},\bar{x}) = -\frac{1}{(1+\bar{x}^2)} = -1, \quad f_w(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1+\bar{x}^2)^2} = 1.$$

The linearized equation of Eq. (4) about $\bar{x}$ is

$$y_{n+1} - y_{n-6} + y_{n-5} - y_n = 0. \tag{5}$$

It follows from Theorem 1 that Eq. (5) is not asymptotically stable. The proof is complete.

For confirming the results of this section, we consider numerical example for $x_{-6} = 1, x_{-5} = 2, x_{-4} = 3, x_{-3} = 4, x_{-2} = 5, x_{-1} = 6, x_0 = 7$,(See Figure 1).
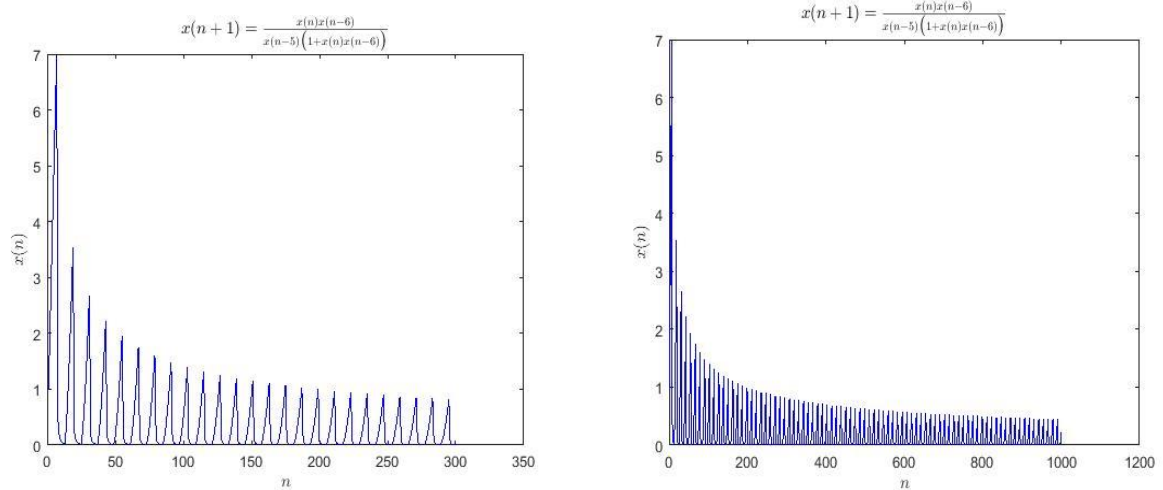
Figure 1

3 Second Case: On the Difference Equation $x_{n+1} = \dfrac{x_n x_{n-6}}{x_{n-5}(-1+x_n x_{n-6})}$

In this subsection we study the following special case of Eq. (1):

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}(-1+x_n x_{n-6})}, \quad n = 0,1,\ldots, \tag{6}$$

where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzeros real numbers.

Theorem 4. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (6). Then the solution of Eq. (6) is bounded and periodic of period 12 given by:

$$
\begin{aligned}
&x_{12n-6} = x_{-6}, \quad x_{12n-5} = x_{-5}, \quad\quad\quad x_{12n-4} = x_{-4}, \quad x_{12n-3} = x_{-3}, \\
&x_{12n-2} = x_{-2}, \quad x_{12n-1} = x_{-1}, \quad\quad\quad x_{12n} = x_0, \quad\quad x_{12n+1} = \frac{x_0 x_{-6}}{x_{-5}(-1+x_0 x_{-6})}, \\
&x_{12n+2} = \frac{x_0 x_{-6}}{x_{-4}}, \quad x_{12n+3} = \frac{x_0 x_{-6}}{x_{-3}(-1+x_0 x_{-6})}, \quad x_{12n+4} = \frac{x_0 x_{-6}}{x_{-2}}, \quad x_{12n+5} = \frac{x_0 x_{-6}}{x_{-1}(-1+x_0 x_{-6})}.
\end{aligned}
$$

 Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n - 1$. That is;

$$
\begin{aligned}
&x_{12n-18} = x_{-6}, \quad x_{12n-17} = x_{-5}, \quad\quad\quad x_{12n-16} = x_{-4}, \quad x_{12n-15} = x_{-3}, \\
&x_{12n-14} = x_{-2}, \quad x_{12n-13} = x_{-1}, \quad\quad\quad x_{12n-12} = x_0, \quad\quad x_{12n-11} = \frac{x_0 x_{-6}}{x_{-5}(-1+x_0 x_{-6})}, \\
&x_{12n-10} = \frac{x_0 x_{-6}}{x_{-4}}, \quad x_{12n-9} = \frac{x_0 x_{-6}}{x_{-3}(-1+x_0 x_{-6})}, \quad x_{12n-8} = \frac{x_0 x_{-6}}{x_{-2}}, \quad x_{12n-7} = \frac{x_0 x_{-6}}{x_{-1}(-1+x_0 x_{-6})}.
\end{aligned}
$$

Now it follows from Eq. (6) that

$$x_{12n-6} = \frac{x_{12n-7}x_{12n-13}}{x_{12n-12}(-1+x_{12n-7}x_{12n-13})} = \frac{\frac{x_0 x_{-6}}{x_{-1}(-1+x_0 x_{-6})}x_{-1}}{x_0\left(-1+\frac{x_0 x_{-6}}{x_{-1}(-1+x_0 x_{-6})}x_{-1}\right)}$$

$$= \frac{\frac{x_{-6}}{(-1+x_0 x_{-6})}}{\left(-1+\frac{x_0 x_{-6}}{(-1+x_0 x_{-6})}\right)} = \frac{\frac{x_{-6}}{(-1+x_0 x_{-6})}}{\left(\frac{1}{(-1+x_0 x_{-6})}\right)} = x_{-6}$$

Similarly

$$x_{12n-5} = \frac{x_{12n-6}x_{12n-12}}{x_{12n-11}(-1+x_{12n-6}x_{12n-12})} = \frac{x_{-6}x_0}{\frac{x_0 x_{-6}}{x_{-5}(-1+x_0 x_{-6})}(-1+x_{-6}x_0)} = x_{-5}$$

$$x_{12n-4} = \frac{x_{12n-5}x_{12n-11}}{x_{12n-10}(-1+x_{12n-5}x_{12n-11})} = \frac{x_{-5}\frac{x_0 x_{-6}}{x_{-5}(-1+x_0 x_{-6})}}{\frac{x_0 x_{-6}}{x_{-4}}\left(-1+x_{-5}\frac{x_0 x_{-6}}{x_{-5}(-1+x_0 x_{-6})}\right)}$$

$$= \frac{\frac{1}{(-1+x_0 x_{-6})}}{\frac{1}{x_{-4}}\left(-1+\frac{x_0 x_{-6}}{(-1+x_0 x_{-6})}\right)} = \frac{\frac{x_{-4}}{(-1+x_0 x_{-6})}}{\left(\frac{1}{(-1+x_0 x_{-6})}\right)} = x_{-4}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 5. Eq. (6) has tree equilibrium points which are $0, \pm\sqrt{2}$ and these equilibrium points are not locally asymptotically stable.

Proof. As the proof of Theorem 3 and will be omitted.

For confirming the results of this section, we consider numerical example for $x_{-6} = 1, x_{-5} = 2, x_{-4} = 3, x_{-3} = 4, x_{-2} = 5, x_{-1} = 6, x_0 = 7$, (See Figure 2).
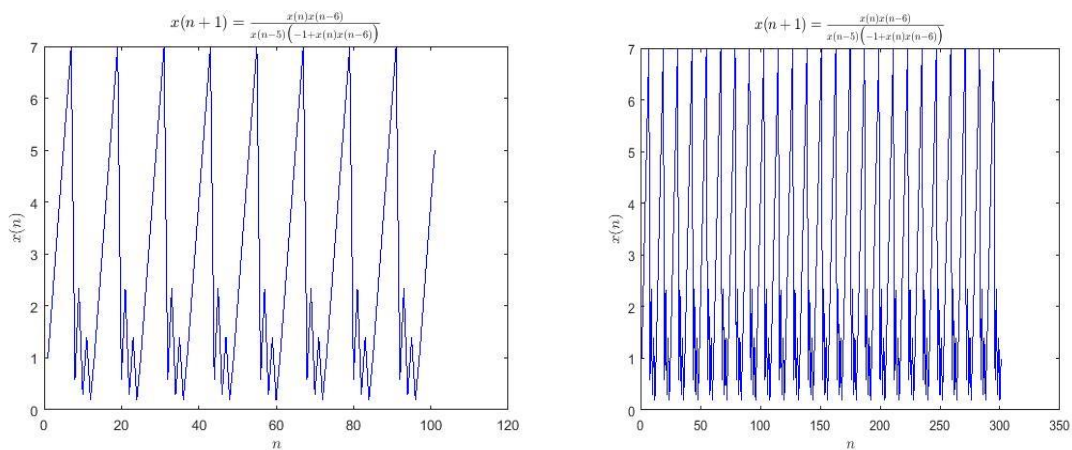

Figure 2

4 Conclusion

This paper discussed local stability, the solutions of some special cases of Eq. (1) and gave numerical examples of each case.

Acknowledgements

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