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Research Article

Fractional Integral and Beta Transform Formulas for the extended Appell-Lauricella Hypergeometric Functions

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Abstract: The fractional integral formulas involving many special functions have been investigated in the literature by many authors due to their application point of view. In the present paper, we aim to establishing some (presumably) new fractional integral formulas a for the new extended Appells and Lauricells type hypergeometric functions introduced by Agarwal *et al.* [3]. As well as, we also establish a Beta transform formula for the extended Appells type hypergeometric function of second kind.

Keywords: Fractional integral operators, Kampé de Fériet Function, Generalized hypergeometric function, Beta transform, Pochhammer symbol, Gamma function.

1.Introduction and Preliminaries

Throughout the paper, let $\mathbb{N}, \mathbb{R}^{+}, \mathbb{C}$ denotes the sets of positive integers, real numbers and complex numbers, respectively.

The Riemann-Liouville fractional integral I_{a+}^{θ} (see, e.g., [20])

$$\left(I_{u+}^{\theta} f\right)(\phi)=\frac{1}{\Gamma(\theta)} \int_{u}^{\phi} \frac{f(\chi)}{(\phi-\chi)^{1-\theta}} d\chi, (\theta \in \mathbb{C}; \Re(\theta)>0) \tag{1.1}$$

where $[\phi]$ means the greatest integer not exceeding real ϕ. Recently, Agarwal *et al.*[3] introduced the following extended hypergeometric functions of two and three variables as follows:

$$F_{1}^{(\alpha,\beta;m)}[a, b, c; d; x, y; p; m]=\sum_{r,s}^{\infty} \frac{B_{p}^{(\alpha,\beta;m)}(a+r+s, d-a)(b)_{r}(c)_{s}}{B(a, d-a)} \frac{x^{r}}{r!} \frac{y^{s}}{s!}, \tag{1.2}$$

$$(\max\{|x|,|y|\}<1; \Re(p)>0)$$

$$F_{2}^{(\alpha,\beta,\alpha\prime,\beta\prime;m)}[a, b, c; d, e; x, y; p; m]=\sum_{r,s}^{\infty} \frac{(a)_{r+s} B_{p}^{(\alpha,\beta;m)}(b+r, d-b) B_{p}^{(\alpha\prime,\beta\prime;m)}(c+s, e-c)}{B(b, d-b) B(c, e-c)} \frac{x^{r}}{r!} \frac{y^{s}}{s!}, \tag{1.3}$$

$$(|x|+|y|<1; \Re(p)>0)$$

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$$F_{D,p}^{(3;\alpha,\beta;m)}[a,b,c,d;e;x,y,z;p;m]=\sum_{r,s,w}^{\infty}\frac{B_p^{(\alpha,\beta;m)}(a+r+s+w,e-a)}{B(a,e-a)}(b)_r(c)_s(d)_w\frac{x^r}{r!}\frac{y^s}{s!}\frac{z^w}{w!}, \tag{1.4}$$

$$(\max\{|x|,|y|,|z|\}<1;\Re(p)>0)$$

where, $B_p^{(\alpha,\beta;m)}(.,.)$ is the generalized Beta function given by (see, for example, [12])

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^m\,(1-t)^m}\right)dt, \tag{1.5}$$

$$(\Re(p),\Re(m)>0)$$

If $p=0$, it obviously reduces to the usual generalized Euler's beta function.

Since last four decades, several extensions of the family of special functions (such as Gamma function, Beta function and generalized hypergeometric functions) have been considered by many researchers (see, e.g., [12, 13]) and their relation with fractional calculus (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). Here by motivated these works, we aim to present some fractional integral formulas for the extended Appell's hypergeometric functions of two variables and Lauricella's hypergeometric function of three variables as well as a Beta transform formula for the extended Appells type hypergeometric function of second kind..

2 .Fractional Integral

In this section, we present certain integral formulas for the generalized Appells type hypergeometric functions and the generalized extended Lauricella's hypergeometric function defined by (1.2), (1.3) and (1.4), respectively.

Theorem 1. *The following fractional integral formulas holds true:*

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1}F_1^{(\alpha,\beta;m)}[a,\mu+\nu,c;d;\psi(t-\tau),y;p;m]\right\}\right)(\theta)$$
$$=(\theta-\tau)^{\mu+\nu-1}\frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}F_1^{(\alpha,\beta;m)}[a,\nu,c;d;\psi\,(\theta-\tau),y;p;m]$$

(2.1)

$$(\tau\in\mathbb{R}^+=[0,\infty);\mu,\nu\in\mathbb{C},\Re(p)>0;\theta>\tau)$$

Proof. In the left hand side of (2.1), using (1.1) and (1.2) and applying term-by-term fractional integration by virtue of the well known formula (see, e.g., [20])

$$\left(I_{a+}^{\nu}\{(t-a)^{\mu-1}\}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(x-a)^{\mu+\nu-1},(\mu,\nu\in\mathbb{C};\Re(\mu)>0,\Re(\nu)>0). \tag{2.2}$$

we have for $\theta>\tau$

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1}F_1^{(\alpha,\beta;m)}[a,\mu+\nu,c;d;\psi(t-\tau),y;p;m]\right\}\right)(\theta)$$
$$=\sum_{r,s}^{\infty}\frac{B_p^{(\alpha,\beta;m)}(a+r+s,d-a)(\mu+\nu)_r(c)_s}{B(a,d-a)}\frac{\psi^r}{r!}\frac{y^s}{s!}\left(I_{\tau+}^{\mu}\{(t-\tau)^{\nu+r-1}\}\right)(\theta) \tag{2.3}$$

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1}F_1^{(\alpha,\beta;m)}[a,\mu+\nu,c;d;\psi(t-\tau),y;p;m]\right\}\right)(\theta)$$

$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} \sum_{r,s}^{\infty} \frac{B_p^{(\alpha,\beta;m)}(a+r+s,d-a)(\nu)_r(c)_s}{B(a,d-a)} \frac{(\psi(\theta-\tau))^r}{r!} \frac{y^s}{s!} \tag{2.4}$$

Applying (1.2), we get the desired result (2.1).

Theorem 2. *The following fractional integral formulas holds true:*

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1} F_1^{(\alpha,\beta;m)}[a,b,\mu+\nu;d;x,\psi(t-\tau);p;m]\right\}\right)(\theta)$$
$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} F_1^{(\alpha,\beta;m)}[a,b,\nu;d;x,\psi\,(\theta-\tau);p;m] \tag{2.5}$$
$$(\tau \in \mathbb{R}^+ = [0,\infty); \mu,\nu \in \mathbb{C}, \Re(p) > 0; \theta > \tau)$$

Proof. Proof of the Theorem 2, would run parallel to Theorem 1. Therefore, we omit its details.

Theorem 3. *The following relations hold true*

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1} F_2^{(\alpha,\beta,\alpha',\beta';m)}[\mu+\nu,b,c;d,e;\psi\,(t-\tau),\xi\,(t-\tau);p;m]\right\}\right)(\theta)$$
$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} F_2^{(\alpha,\beta,\alpha',\beta';m)}[\nu,b,c;d,e;\psi\,(\theta-\tau),\xi\,(\theta-\tau);p;m] \tag{2.6}$$
$$(\tau \in \mathbb{R}^+ = [0,\infty); \mu,\nu \in \mathbb{C}, \Re(p) > 0; \theta > \tau)$$

Proof. By using (1.3) and (1.1), a similar argument as in the proof of Theorem 1 and Theorem 2, we will reached at (2.6). So, details are omitted.

On the same way, we find following Theorems involving the Lauricella's hypergeometric function.

Theorem 4. *The following relations hold true*

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1} F_{D,p}^{(3;\alpha,\beta;m)}[a,\mu+\nu,c,d;e;\psi\,(t-\tau),y,z;p;m]\right\}\right)(\phi)$$
$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} F_{D,p}^{(3;\alpha,\beta;p;m)}[a,\nu,c,d;e;\psi\,(\theta-\tau),y,z;p;m] \tag{2.7}$$

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1} F_{D,p}^{(3;\alpha,\beta;m)}[a,b,\mu+\nu,d;e;x,\psi\,(t-\tau),z;p;m]\right\}\right)(\phi)$$
$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} F_{D,p}^{(3;\alpha,\beta;p;m)}[a,b,\nu,d;e;x,\psi\,(\theta-\tau),z;p;m] \tag{2.8}$$

$$\left(I_{\tau+}^{\mu}\left\{(t-\tau)^{\nu-1} F_{D,p}^{(3;\alpha,\beta;m)}[a,b,c,\mu+\nu;e;x,y,\psi\,(t-\tau);p;m]\right\}\right)(\phi)$$
$$= (\theta-\tau)^{\mu+\nu-1} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} F_{D,p}^{(3;\alpha,\beta;p;m)}[a,b,c,\nu;e;x,y,\psi\,(\theta-\tau);p;m] \tag{2.9}$$
$$(\tau \in \mathbb{R}^+ = [0,\infty); \mu,\nu \in \mathbb{C}, \Re(p) > 0; \theta > \tau)$$

Proof. Applying (1.4) and (1.1), and a similar argument as in the proof of Theorem 1, we establish the (2.7), (2.8) and (2.9), respectively. This complete the proof of Theorem 4.

3. Beta transforms

Here, we establish certain interesting Beta transform associated with generalized hypergeometric functions of two variable $F_2^{(\alpha,\beta,\alpha',\beta';m)}[.]$.

We recall the Beta transform of $f(z)$ defined by (see [21])

$$B\{f(z):a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\,dz. \tag{3.1}$$

Theorem 5 *Let $\Re(p) > 0$ and $u, v \in \mathbb{C}$ with $\Re(u) > 0$ and $\Re(v) > 0$, then the following relations hold true:*

$$B\left\{F_2^{(\alpha,\beta,\alpha\prime,\beta\prime;m)}[u+v,b,c;d,e;x\,z,y\,z;p;m];u,v\right\} = \frac{\Gamma(v)}{\Gamma(u+v)}\,F_2^{(\alpha,\beta,\alpha\prime,\beta\prime;m)}[u,b,c;d,e;x,y;p;m] \quad (3.2)$$

Proof. Let $\mathcal{B}$ be the left-hand side of (3.2). Using the definition of Beta transform, we have

$$\mathcal{B} = \int_0^1 z^{u-1}(1-z)^{v-1}\left(F_2^{(\alpha,\beta,\alpha\prime,\beta\prime;m)}[u+v,b,c;d,e;x\,z,y\,z;p;m]\right)dz, \quad (3.3)$$

which, using (1.2) and changing the order of integration and summation, which is valid under the conditions of Theorem 5, yields

$$\mathcal{B} = \sum_{r,s}^{\infty} \frac{(u+v)_{r+s}B_p^{(\alpha,\beta;m)}(b+r,d-b)B_p^{(\alpha\prime,\beta\prime;m)}(c+s,e-c)}{B(b,d-b)B(c,e-c)}\frac{x^r}{r!}\frac{y^s}{s!}\left\{\int_0^1 z^{u+r+s-1}(1-z)^{v-1}\,dz\right\}. \quad (3.4)$$

Applying the beta function $B(\alpha,\beta)$ defined by (see, e.g., [19, Section 1.1])

$$B(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta \in \mathbb{C}\backslash\mathbb{Z}_0^-), \end{cases} \quad (3.5)$$

to (3.4), we get

$$\mathcal{B} = \frac{\Gamma(v)}{\Gamma(u+v)}\sum_{r,s}^{\infty} \frac{(u)_{r+s}B_p^{(\alpha,\beta;m)}(b+r,d-b)B_p^{(\alpha\prime,\beta\prime;m)}(c+s,e-c)}{B(b,d-b)B(c,e-c)}\frac{x^r}{r!}\frac{y^s}{s!}, \quad (3.6)$$

and by using (1.2), yields the right-hand side of Theorem 5.

4. Concluding Remarks

We conclude our present investigation by remarking that here with the help of the well known Riemann-Liouville fractional integral operator, we have obtained the composition formulas of the R-L integral (1) associated the generalized hypergeometric functions of two and three variables. Also, we presented Beta-transform formula, all the results presented here, may be very useful in the study of engineering problems like boundary value problems, statistic theory and computer science.

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References

[1] Agarwal, P. (2017). "Some inequalities involving Hadamard type k-fractional integral operators", Mathematical Methods in the Applied Sciences, 40(11), 3882-3891.

[2] Agarwal, P., Al-Mdallal, Q., Cho, YJ., Jain, S. (2018). "Fractional differential equations for the generalized Mittag-Leffler function", Advances in Difference Equations, 58

[3] Agarwal, P., Choi, J., Jain, S. (2015). “Extended hypergeometric functions of two and three variables”, Commun. Korean Math. Soc. 30 (4), 403-414.

[4] Agarwal, P., Jleli, M., Tomar, M. (2017). "Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals" Journal of inequalities and applications, 55.

[5] Agarwal, P., Jain, S., Mansour, T. (2017). "Further extended Caputo fractional derivative operator and its applications" Russian Journal of Mathematical Physics, 24(4), 415-425.

[6] Agarwal, P., Nieto, JJ., Luo, MJ. (2017). "Extended Riemann-Liouville type fractional derivative operator with applications", Open Mathematics, 15(1), 1667-1681

[7]Agarwal, P., Al-Mdallal, Q., Cho, YJ., Jain, S. (2018), "Fractional differential equations for the generalized Mittag-Leffler function", Advances in Difference Equations, 58

[8] Jain, S., Agarwal, P. Kıymaz, İO., Çetinkaya, A. (2018). "Some composition formulae for the MSM fractional integral operator with the multi-index Mittag-Leffler functions", AIP Conference Proceedings, 1926(1).

[9] Kilbas, AA., Srivastava, HM., Trujillo, JJ. (2006). “Theory and Applications of Fractional Differential Equations”, North-Holland Mathematics Studies 204, Elsevier, Amsterdam.

[10] Kiryakova, V. (1994). “Generalized Fractional Calculus and Applications”, Longman Scientific & Tech., Essex.

[11] Kiryakova, V. (2008). “A brief story about the operators of the generalized fractional calculus”, Fract. Calc. Appl. Anal. 11 (2), 203-220.

[12] Lee, DM., Rathie, AK., Parmar, RK., Kim, YS. (2011). “Generalization of extended Beta function, hypergeometric and confluent hypergeometric functions”, Honam Math. J. 33(2), 187-206.

[13] Luo, MJ., Milovanovic, GV., Agawal, P.(2014). “Some results on the extended beta and extended hypergeometric functions”, Appl. Math. Comput. 248, 631-651.

[14] Marichev, IO. (1974). “Volterra equation of Mellin convolution type with a Horn function in the kernel (In Russian)”, Izv. ANBSSR Ser. Fiz.-Mat. Nauk, 1, 128–129.

[15] McBride, AC. (1982). “Fractional powers of a class of ordinary differential operators”, Proc. London Math. Soc. (III) 45, 519-546.

[16] Saigo, M. (1977/78). “A remark on integral operators involving the Gauss hypergeometric functions”, Math. Rep. Kyushu Univ. 11(2), 135–143.

[17] Saigo, M., Maeda, N. (1996). “More generalization of fractional calculus, Transform Methods and Special Functions”, Varna, Bulgaria, 386-400.

[18] Srivastava, HM., Agawal, P., Jain, S. (2014). “Generating functions for the generalized Gauss hypergeometric functions”, Appl. Math. Comput. 247, 348-352.

[19] Srivastava, HM., Choi, J. (2012). “Zeta and q-Zeta Functions and Associated Series and Integrals”, Elsevier Science Publishers, Amsterdam, London and New York.

[20] Samko, SG., Kilbas, AA., Marichev, OI. (1993). “Fractional Integrals and Derivatives. Theory and Applications”, Gordon and Breach, Yverdon *et al*.

[21] Sneddon, IN. (1979). “The Use of Integral Transforms”, Tata McGraw-Hill, New Delhi.