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Notes on Weighted Mean and Median

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Abstract: The paper investigates the application and generalization of the weighted mean and median by using continuous and convex functions. The paper offers a clear and systematic approach to the notion of weighted medians. As a result, essential characteristics of weighted medians are presented better.

Keywords: weighted mean, weighted median, global minimum.

1. Introduction

Let $n \geq 2$ be an integer, let $x_1, \ldots, x_n \in \mathbb{R}$ be points, and let $w_1, \ldots, w_n \in [0,1]$ be coefficients satisfying $\sum_{i=1}^{n} w_i = 1$, as such are usually called weights.

An arithmetic mean of the given points is the inserted point

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{1}$$

A weighted (arithmetic) mean of the given points respecting the given weights is the inserted point

$$\bar{x} = \sum_{i=1}^{n} w_i x_i. \tag{2}$$

If the points x_i are sorted from smaller to larger, we are able to observe medians. Suppose that $x_1 < \ldots < x_n$. If n is odd as $n = 2k-1$, then the middle point x_k is a median. If n is even as $n = 2k$, then the middle points x_k and x_{k+1} are a lower and upper median. Weighted medians can be defined as follows.

If for a weight w_k, where $k \in \{1, \ldots, n\}$, applies

$$\sum_{i=1}^{k} w_i > \frac{1}{2} \text{ and } \sum_{i=k}^{n} w_i > \frac{1}{2}, \tag{3}$$

then the point x_k is a weighted median. In this case, w_k is unique and positive.

If for weights w_k and w_l, where $k \in \{1, \ldots, n-1\}$ and $l \in \{k+1, \ldots, n\}$, apply

$$\sum_{i=1}^{k} w_i = \frac{1}{2} \text{ and } \sum_{i=l}^{n} w_i = \frac{1}{2} \tag{4}$$

with the integers k and l as the smallest and largest possible, then the points x_k and x_l are a lower and upper weighted median. In this case, w_k and w_l are unique and positive, and if $l \geq k+2$, then $w_{k+1} = \ldots = w_{l-1} = 0$.

The conditions in equation (3) and equation (4) exclude each other. If all weights are the same,

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$w_1 = \ldots = w_n = 1/n$, then the weighted arithmetic mean coincides with arithmetic mean, and the weighted medians coincide with medians.

2. Weighted Mean and Median as the Minimum Points

In this section, we employ a collection of functions defined on $\mathbb{R}$ whose members have a global minimum. Let $x_1, \ldots, x_n \in \mathbb{R}$ be points, let $a_1, \ldots, a_n \geq 0$ be coefficients of which at least one is positive, let $p \geq 1$ and $q > 0$ be exponents, and let

$$f(x) = \left(\sum_{i=1}^{n} a_i \,|\, x - x_i \,|^p \right)^{q/p}. \tag{5}$$

Including the limit case by letting p tends to infinity, we get

$$f(x) = (\max\{a_1 \,|\, x - x_1 \,|, \ldots, a_n \,|\, x - x_n \,|\})^q. \tag{6}$$

This limit is a consequence of the transition from the p-norm to *max*-norm, see equations (9) and (10). The above functions are continuous and satisfy $\lim_{|x|\to\infty} f(x) = \infty$, so they have a global minimum. The related functions $g = f^{1/q}$ are convex. The minimum points of f and g coincide because a power function with a positive exponent is increasing on the interval of nonnegative numbers. When $q \geq 1$, the functions in equations (5) and (6) are convex as the compositions of increasing convex and convex functions ($f = h \circ g$, where $h(x) = x^q$ for $x \geq 0$). If $p > 1$ and $q > 1$, the observed functions are strictly convex.

We point out the basic topological properties of convex functions. Let $I^* \subseteq I \subseteq \mathbb{R}$ be intervals with the nonempty interior, and let $f : I \to \mathbb{R}$ be a convex function. Then f is continuous on the interior of I, and each its local minimum is global. As for the minimum, if $x^* \in I^*$ is a point so that $f(x^*)$ is a minimum value on I^*, then $f(x^*)$ is minimum value on I.

Let $c \in \mathbb{R}$ be a coefficient, let $p > 1$ be an exponent, and let $h(x) = |\, x - c \,|^p$. The function h is strictly convex, and differentiable at each point $x \in \mathbb{R}$ with the derivative

$$h'(x) = \begin{cases} p \dfrac{|\, x - c \,|^p}{x - c} & \text{if} \quad x \neq c \\ 0 & \text{if} \quad x = c \end{cases}.$$

The derivative of the convex function $h(x) = |\, x - c \,|$ is included in first line of the above equation with $p = 1$.

In what follows, we are discussing global minimum points of the functions in equations (5) and (6). The interval which includes the given points x_i comes into play. If we denote

$$x_{(1)} = \min\{x_1, \ldots, x_n\} \text{ and } x_{(n)} = \max\{x_1, \ldots, x_n\},$$

then the closed interval $[x_{(1)}, x_{(n)}]$ contains the global minimum points of the above functions. Further, it all depends on the exponent p.

A strict global minimum point exists in the case $p > 1$.

Lemma 2.1. *Let $x_1, \ldots, x_n$ be points of $\mathbb{R}$, let $a_1, \ldots, a_n \geq 0$ be coefficients such that $a = \sum_{i=1}^{n} a_i > 0$, let $p > 1$ and $q > 0$ be exponents, and let*

$$f(x) = \left(\sum_{i=1}^{n} a_i \,|\, x - x_i \,|^p \right)^q.$$

Then a strict global minimum point of f exists in the interval $[x_{(1)}, x_{(n)}]$.

Proof. We are looking for the unique minimum point of the strictly convex function

$$g(x)=(f(x))^{1/q}=\sum_{i=1}^{n}a_i\,|x-x_i|^p\,.$$

Using the fact that at least one of the coefficients a_i is positive, and applying the derivative

$$g'(x)=p\sum_{i=1}^{n}a_i\frac{|x-x_i|^p}{x-x_i}$$

outside the interval $[x_{(1)},x_{(n)}]$, it follows that $g'(x)<0$ if $x\in(-\infty,x_{(1)})$, and $g'(x)>0$ if $x\in(x_{(n)},+\infty)$. The strictly convex function g decreases on $(-\infty,x_{(1)})$ and increases on $(x_{(n)},+\infty)$, and so it reaches a global minimum at a unique point of the remaining part $[x_{(1)},x_{(n)}]$. The same is true for the function f. □

The function f in equation (6) also has a strict global minimum point in the interval $[x_{(1)},x_{(n)}]$.

The weighted mean plays a role in the case $p=2$.

Lemma 2.2. **1***Let* $x_1,\ldots,x_n$ *be points of* $\mathbb{R}$, *let* $a_1,\ldots,a_n\geq 0$ *be coefficients such that* $a=\sum_{i=1}^{n}a_i>0$, *let* $q>0$ *be an exponent, and let*

$$f(x)=\left(\sum_{i=1}^{n}a_i\,|x-x_i|^2\right)^q.$$

Then a strict global minimum point of f *exists as the weighted mean of the points* x_i *respecting the weights* $w_i=a_i/a$.

Proof. Using the sums

$$b=\sum_{i=1}^{n}a_ix_i \text{ and } c=\sum_{i=1}^{n}a_ix_i^2$$

as the coefficients in the presentation

$$g(x)=(f(x))^{1/q}=ax^2-2bx+c=a\left(x-\frac{b}{a}\right)^2+\frac{ac-b^2}{a},$$

we can conclude that the unique minimum point of the functions g and f is

$$\bar{x}=\frac{b}{a}=\frac{1}{a}\sum_{i=1}^{n}a_ix_i=\sum_{i=1}^{n}w_ix_i,$$

representing the weighted mean of the points x_i respecting the weights w_i. □

The weighted medians occur in the case $p=1$.

Lemma 2.3. **2***Let* $x_1<\ldots<x_n$ *be strictly ordered points of* $\mathbb{R}$, *let* $a_1,\ldots,a_n\geq 0$ *be coefficients such that* $a=\sum_{i=1}^{n}a_i>0$, *let* $q>0$ *be an exponent, and let*

$$f(x)=\left(\sum_{i=1}^{n}a_i\,|x-x_i|\right)^q.$$

If the point x_k *exists as a weighted median of the points* x_i *respecting the weights* $w_i=a_i/a$, *then* x_k *is a strict global minimum point of* f.

If the points x_k and x_l exist as a lower and upper weighted median, then each element of $[x_k, x_l]$ is a global minimum point of f.

Proof. We include the weights w_i through the function

$$g(x)=\frac{1}{a}(f(x))^{1/q}=\sum_{i=1}^{n}w_i\,|x-x_i|,$$

which is convex and so continuous on $\mathbb{R}$, and differentiable on $\mathbb{R}\setminus\{x_1,\ldots,x_n\}$ with the derivative

$$g'(x)=\sum_{i=1}^{n}w_i\frac{|x-x_i|}{x-x_i}.$$

Obviously, $g'(x)=-\sum_{i=1}^{n}w_i=-1$ if $x<x_1$, and $g'(x)=\sum_{i=1}^{n}w_i=1$ if $x>x_n$. Let δ be a positive number so that for each point x_k the interval $(x_k-\delta, x_k+\delta)$ does not contain any point x_i other than x_k.

The derivative of the function g around a point x_k stands as

$$g'(x)=\begin{cases}\sum_{i=1}^{k-1}w_i-\sum_{i=k}^{n}w_i & \text{if} \quad x\in(x_k-\delta,x_k)\\ \sum_{i=1}^{k}w_i-\sum_{i=k+1}^{n}w_i & \text{if} \quad x\in(x_k,x_k+\delta)\end{cases}. \tag{7}$$

Suppose that x_k is a weighted median. Applying the conditions in formula (3) to the derivative in formula (7), we obtain that $g'(x)<0$ if $x\in(x_k-\delta,x_k)$, and $g'(x)>0$ if $x\in(x_k,x_k+\delta)$. Since g is continuous, x_k is a strict minimum point on $(x_k-\delta_k, x_k+\delta_k)$. Since g is convex, x_k is also a strict minimum point on $\mathbb{R}$. Thus, x_k is a strict global minimum point of the functions g and f.

The derivative of the function g around points x_k and x_l, where $l\geq k+1$ and $w_{k+1}=\ldots=w_{l-1}=0$ if $l\geq k+2$, stands as

$$g'(x)=\begin{cases}\sum_{i=1}^{k-1}w_i-w_k-\sum_{i=l}^{n}w_i & \text{if} \quad x\in(x_k-\delta,x_k)\\ \sum_{i=1}^{k}w_i-\sum_{i=l}^{n}w_i & \text{if} \quad x\in(x_k,x_l)\\ \sum_{i=1}^{k}w_i+w_l-\sum_{i=l+1}^{n}w_i & \text{if} \quad x\in(x_l,x_l+\delta)\end{cases}. \tag{8}$$

Suppose that x_k and x_l are a lower and upper weighted median. Applying the conditions in formula (4) to the derivative in formula (8), we obtain that $g'(x)<0$ if $x\in(x_k-\delta,x_k)$, $g'(x)=0$ if $x\in(x_k,x_l)$, and $g'(x)>0$ if $x\in(x_l,x_l+\delta)$. Each element of $[x_k,x_l]$ is a minimum point of g on $(x_k-\delta, x_l+\delta)$, and consequently on $\mathbb{R}$. Thus, each such element is a global minimum point of g and f. □

Remark 2.4. 3*If at least two of the points x_i are distinct, the functions used in the above lemmas are positive. Assuming that $q<0$, a global maximum point appears instead of global minimum point.*

In the last decades, weighted median algorithms and filters are widely used in the science, engineering and economics. Three algorithms for the weighted median problem are presented in [1]. Improved performances of the weighted median filter for the image processing are presented in [2].

3. Main Results

The generalization of the previous section will be done in the space $\mathbb{R}^m$ assuming that $m \geq 1$, and we will utilize a collection of functions defined on $\mathbb{R}^m$ that have a global minimum.

Basic norms on the space $\mathbb{R}^m$ are p-norms generated by real numbers $p \geq 1$. The p-norm of a point $T = (x_1, \ldots, x_m) \in \mathbb{R}^m$ is

$$\| T \|_p = \left(\sum_{i=1}^{m} | x_i |^p \right)^{1/p}, \tag{9}$$

and the *max*-norm as the limit case when p tends to infinity is

$$\| T \|_\infty = \max\{| x_1 |, \ldots, | x_m |\}. \tag{10}$$

Let $T_1, \ldots, T_n \in \mathbb{R}^m$ be points, let $a_1, \ldots, a_n \geq 0$ be coefficients of which at least one is positive, let $p \geq 1$ and $q > 0$ be exponents, and let

$$f(T) = \left(\sum_{i=1}^{n} a_i \| T - T_i \|_p^p \right)^{q/p}. \tag{11}$$

Sending p to infinity, it follows that

$$f(T) = (\max\{a_1 \| T - T_1 \|_\infty, \ldots, a_n \| T - T_n \|_\infty\})^q. \tag{12}$$

The above functions are continuous and satisfy $\lim_{\|T\| \to \infty} f(T) = \infty$ for every norm on $\mathbb{R}^m$, thus each one of them attains a global minimum value.

Problems relating to the global minimum of convex functions can be found in [3, chapter The Variational Approach of PDE]. Some algorithms for calculating extremum points of convex functions are presented in [5]. Optimization problems concerning convex functions are discussed in [4].

Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ be points in $\mathbb{R}^m$, and let $\preceq$ be the partial order relation of points in $\mathbb{R}^m$ stated by

$$A \preceq B \text{ if and only if } a_1 \leq b_1, \ldots, a_m \leq b_m.$$

The above relation generates the closed interval between A and B in the form of the m-fold Cartesian product of the closed intervals $[a_j, b_j]$,

$$\preceq A, B \succeq = [a_1, b_1] \times \ldots \times [a_m, b_m].$$

Throughout the section, the coordinates of the points T_i ($i = 1, \ldots, n$) will be denoted with

$$T_i = (x_{i1}, \ldots, x_{im}),$$

the smallest and largest of the j-coordinates ($j = 1, \ldots, m$) of the points T_i with

$$x_{(1)j} = \min\{x_{1j}, \ldots, x_{nj}\} \text{ and } x_{(n)j} = \max\{x_{1j}, \ldots, x_{nj}\},$$

and the points with the smallest and largest coordinates of the points T_i with

$$T_{(1)} = (x_{(1)1}, \ldots, x_{(1)m}) \text{ and } T_{(n)} = (x_{(n)1}, \ldots, x_{(n)m}).$$

Theorem 3.1. 4*Let $T_1, \ldots, T_n$ be points of $\mathbb{R}^m$, let $a_1, \ldots, a_n \geq 0$ be coefficients such that $a = \sum_{i=1}^{n} a_i > 0$, let $p > 1$ and $q > 0$ be exponents, and let*

$$f(T)=\left(\sum_{i=1}^{n}a_i\,\|T-T_i\|_p^p\right)^q.$$

Then a strict global minimum point of f *exists in the interval* $\preceq T_{(1)},T_{(n)}\succeq$.

Proof. The function $g=f^{1/q}$ is strictly convex. Its coordinate representation

$$g(x_1,\dots,x_m)=\sum_{i=1}^{n}\sum_{j=1}^{m}a_i\,|x_j-x_{ij}|^p$$

can be presented as the sum of the one variable strictly convex functions

$$g_j(x)=\sum_{i=1}^{n}a_i\,|x-x_{ij}|^p$$

in the form of

$$g(x_1,\dots,x_m)=\sum_{j=1}^{m}g_j(x_j). \tag{13}$$

The function g_j has a strict global minimum point in the interval $[x_{(1)j},x_{(n)j}]$ by Lemma 2.1. Considering equation (13), the function g has a strict global minimum point in the m-fold product $[x_{(1)1},x_{(n)1}]\times\dots\times[x_{(1)m},x_{(n)m}]=\preceq T_{(1)},T_{(n)}\succeq$. The latter also applies to the function f. □

The function f in equation (12) also has a strict global minimum point in the interval $\preceq T_{(1)},T_{(n)}\succeq$.

A special case of the above theorem for $p=2$ specifically provides a strict global minimum point as the weighted median.

Theorem 3.2. 5*Let* $T_1,\dots,T_n$ *be points of* $\mathbb{R}^m$, *let* $a_1,\dots,a_n\geq 0$ *be coefficients such that* $a=\sum_{i=1}^{n}a_i>0$, *let* $q>0$ *be an exponent, and let*

$$f(T)=\left(\sum_{i=1}^{n}a_i\,\|T-T_i\|_2^2\right)^q.$$

Then a strict global minimum point of f *exists as the weighted mean of the points* T_i *respecting the weights* $w_i=a_i/a$.

Proof. Putting $g=f^{1/q}$, we can use equation (13) with the functions

$$g_j(x)=\sum_{i=1}^{n}a_i\,|x-x_{ij}|^2.$$

Similarly as in the proof of Lemma 2.2, including the sums

$$b_j=\sum_{i=1}^{n}a_ix_{ij}\ \text{and}\ c_j=\sum_{i=1}^{n}a_ix_{ij}^2,$$

it follows that

$$g_j(x)=a\left(x-\frac{b_j}{a}\right)^2+\frac{ac_j-b_j^2}{a},$$

which indicates that the function g_j has the unique minimum point

$$\bar{x}_j=\frac{b_j}{a}=\frac{1}{a}\sum_{i=1}^{n}a_ix_{ij}=\sum_{i=1}^{n}w_ix_{ij}.$$

According to equation (13), the unique minimum point of the function g, and consequently the function $f=g^q$, is

$$\overline{T} = (\overline{x}_1,\dots,\overline{x}_m) = \sum_{i=1}^{n} w_i (x_{i1},\dots,x_{im}) = \sum_{i=1}^{n} w_i T_i,$$

as the weighted mean of the points T_i respecting the weights w_i . □

In the next theorem, we will use the strict partial order relation of points in $\mathbb{R}^m$ by means of

$$A \prec B \text{ if and only if } a_1 < b_1,\dots,a_m < b_m.$$

Employing the partial and strict partial order relation, we can generalize Lemma 2.3. The case $p = 1$ for multivariate functions also refers to weighted medians.

Theorem 3.3. **6***Let $T_1 \prec \dots \prec T_n$ be strictly ordered points of $\mathbb{R}^m$, let $a_1,\dots,a_n \geq 0$ be coefficients such that $a = \sum_{i=1}^{n} a_i > 0$, let $q > 0$ be an exponent, and let*

$$f(T) = \left(\sum_{i=1}^{n} a_i \| T - T_i \|_1 \right)^q .$$

If the point T_k exists as a weighted median of the points T_i respecting the weights $w_i = a_i / a$, then T_k is a strict global minimum point of f .

If the points T_k and T_l exist as a lower and upper weighted median, then each element of $\preceq T_k, T_l \succeq$ is a global minimum point of f .

Proof. We can consider the related function $g = (1/a) f^{1/q}$, and use equation (13) with the functions

$$g_j(x) = \sum_{i=1}^{n} w_i | x - x_{ij} | .$$

Since $T_1 \prec \dots \prec T_n$, the points x_{ij} satisfy the strict order

$$x_{1j} < \dots < x_{nj} \tag{14}$$

for every $j = 1,\dots,m$. Therefore, we have the following two convenient cases.

The point x_{kj} is a weighted median of the points x_{ij} respecting the weights w_i. Then x_{kj} is a strict global minimum point of g_j by Lemma 2.3. This applies to each index j. Thus the point T_k is a weighted median of the points T_i respecting the weights w_i, and T_k is a strict global minimum point of g by equation (13).

The points x_{kj} and x_{lj} are a lower and upper weighted median. Then each element of $[x_{kj}, x_{lj}]$ is a global minimum point of g_j by Lemma 2.3. It refers to each j. Thus the points T_k and T_l are a lower and upper weighted median, and each element of $[x_{k1}, x_{l1}] \times \dots \times [x_{km}, x_{lm}] = \preceq T_k, T_l \succeq$ is a global minimum point of g by equation (13). □

Remark 3.4. **7***If at least two of the points T_i are distinct, then using $q < 0$ in the above theorems, we have a global maximum point instead of global minimum point.*

4. Examples

Example 4.1. 8*Find a global minimum value of the function*

$$f(x,y) = (2(|x|^2 + |y-1|^2) + 3(|x-2|^2 + |y|^2) + 4(|x-3|^2 + |y+5|^2))^{5/2}.$$

According to Theorem 3.2, we have to determine the weighted mean of the points

$$T_1 = (0,1), T_2 = (2,0), T_3 = (3,-5)$$

respecting the weights

$$w_1 = \frac{2}{9}, w_2 = \frac{3}{9}, w_3 = \frac{4}{9}.$$

Using equation (2), we get the weighted mean

$$\overline{T} = \sum_{i=1}^{3} w_i T_i = (2,-2)$$

as the unique minimum point, and so the strict global minimum value

$$f(2,-2) = 78^{5/2}.$$

Example 4.2. 9*Find a global minimum value of the function*

$$f(x,y) = (4(|x|+|y|) + 2(|x-2|+|y-2|) + 7(|x-5|+|y-6|))^{1/3}.$$

According to Theorem 3.3, we need to examine the weighted medians of the points

$$T_1 = (0,0) \prec T_2 = (2,2) \prec T_3 = (5,6)$$

respecting the weights

$$w_1 = \frac{4}{13}, w_2 = \frac{2}{13}, w_3 = \frac{7}{13}.$$

Using equation (3), we find the weighted median

$$T_3 = (5,6)$$

as the unique minimum point, and so the strict global minimum value

$$f(5,6) = 58^{1/3}.$$

Example 4.3. 10*Find a global maximum value of the function*

$$f(x,y) = (5(|x+2|+|y|) + 2(|x-3|+|y-1|) + 3(|x-4|+|y-2|))^{-4}.$$

According to Theorem 3.3, we need to examine the weighted medians of the points

$$T_1 = (-2,0) \prec T_2 = (3,1) \prec T_3 = (4,2)$$

respecting the weights

$$w_1 = \frac{5}{10}, w_2 = \frac{2}{10}, w_3 = \frac{3}{10}.$$

Using equation (4), we find that the points

$$T_1 = (-2,0) \text{ and } T_2 = (3,1)$$

are a lower and upper weighted median. Relying on Theorem 3.3 and Remark 3.4, we can take any point $T^* \in \preceq T_1, T_2 \succeq = [-2,3] \times [0,1]$ *to get the global maximum value*

$$f(T^*) = f(-2,0) = 36^{-4}.$$

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