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Research Article

Numerical computing for the moments of the Heston model

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Abstract: The Heston model assumes that the underlying stock price follows a Black-Scholes model, but with stochastic variance that follows CIR model. This model has no general explicit solution. In this paper, by using Feynman-Kac formula and Adomian decomposition method, the moments of solution for the Heston model is computed numerically. The moments have many applications in statistics. For instance, the first moment is the mean, the second moment is the variance, the third moment is the skewness, the fourth moment is the kurtosis, and so forth. Finally, to show the simplicity and efficiency of the proposed method, numerical example is presented.

***Keywords**:* Heston model, Stochastic differential equation, the Moments, Feynman-kac formula, Adomian decomposition method, Parabolic partial differential equation, Convergent series.

1. INTRODUCTION

A Stochastic differential equation (SDE) is a differential equation in which one or more of the terms has random components. In general, SDE has the following form

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t), \qquad (1)$$

where $\mu(X(t),t)$ is drift term, and $\sigma(X(t),t)$ is diffusion term. SDEs are used in biology (for e.g. in the epidemic models, predator-prey models, and population models), physics (for e.g. in the ion transport, nuclear reactor kinetics, chemical reaction, and cotton fiber breakage), and stochastic control to model various phenomena. Also, in finance, SDEs are used to model stock and asset prices, and interest rates. The return of the asset price at time t has two deterministic and random parts. In other words

$$\frac{dS(t)}{S(t)} = deterministic\ part + random\ part \qquad (2)$$

Assuming that the deposit interest rate is $r > 0$, therefore the deterministic part in the relation (2), is rdt. The random part represents the response to external effects, such as unexpected news. There are many external effects so by the well-known central limit theorem, the random part can be represented by a normal distribution with mean zero and variance $v(t)^2dt$. Therefore, the right-hand side of relation (2) can be expressed as

$$\frac{dS(t)}{S(t)} = rdt + N(0, v(t)^2dt) = rdt + v(t)N(0, dt). \qquad (3)$$

$N(0, dt)$ can be replaced by $dW(t)$ ($dW(t)$ is defined in section 2), so the equation (3) changes to

$$\frac{dS(t)}{S(t)} = rdt + v(t)dW(t). \qquad (4)$$

In finance, $v(t)$ is known as the volatility. If the volatility $v(t)$ is independent of the underlying asset price, say

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$v(t) = const.$, the asset price follow the well-known Black-Scholes stochastic differential equation

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t). \tag{5}$$

The solution of this stochastic differential equation is the classical geometric Brownian motion and it is given by

$$S(t) = S(0)exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right). \tag{6}$$

There are various types of volatility functions used in financial modeling. In Theta process $v(t) = \sigma S^{\theta-1}(t)$, hence

$$\frac{dS(t)}{S(t)} = rdt + \sigma S^{\theta-1}(t)dW(t). \tag{7}$$

Heston model assumes that the underlying asset price, $S(t)$, follows a Black-Scholes stochastic differential equation, but the stochastic variance, $v(t)$, that follows the CIR model [6]. Let $\{W(t)^s | t \geq 0\}$ and $\{W(t)^\sigma | t \geq 0\}$ be two standard Wiener process with correlation parameter $\rho \in (-1,1)$. The Heston model is represented by the bivariate system of stochastic differential equations

$$\begin{cases} S(t) = \mu S(t)dt + \sigma(t)S(t)dW^s(t), \\ d\sigma^2(t) = \kappa\left(\theta - \sigma^2(t)\right)dt + \alpha\sigma(t)dW^\sigma(t), \\ dW^s(t)dW^\sigma(t) = \rho dt, \end{cases} \tag{8}$$

where μ, κ, θ, and α are constants, and $\sigma(t)$ is the stochastic volatility of $S(t)$. In the second equation (8), θ is average price volatility and as t tends to infinity, the expected value of $\sigma^2(t)$ tends to θ. κ is rate at which $\sigma^2(t)$ reverts to θ. In addition, α is volatility of volatility and determines the variance of $\sigma^2(t)$.

Feynman-Kac formula named after Richard Feynman and Mark Kac, expresses a close connection between the expectations for solutions of SDEs and partial differential equations (PDEs) [2-5]. Let $\{X(t)\}_{t\geq 0}$ be a solution of the following SDE

$$dX(t) = a(X(t), t)dt + \sigma(X(t), t)dW(t), \tag{9}$$

Assume that f and ρ be given functions. Fix a final time $T > 0$ and define a new function $V(x, t)$ for $t \in [0, T]$ by

$$V(x, t) = e^{-\int_t^T \rho(u)du} E[f(X(T)|X(t) = x]. \tag{10}$$

Assume that $V(x, t) < \infty$ for all (x, t). Then $V(x, t)$ solves the following boundary value problem [2-5]

$$\begin{cases} \frac{dV(x,t)}{dt} + \frac{\sigma^2(x,t)}{2}\frac{d^2V(x,t)}{dx^2} + a(x,t)\frac{dV(x,t)}{dx} = \rho(t)V(x,t), \\ V(x,T) = f(x). \end{cases} \tag{11}$$

Feynman-Kac formula can be extended for multi-dimensional diffusion processes. Let $\{W^i(t)\}_{t\geq 0}$, $i = 1,2, \dots, n$ be a sequence of standard Wiener processes. Let $X^{(i)}(t)$, for $i = 1,2, \dots, n$ be the solution of the following SDE

$$dX^{(i)}(t) = \mu^{(i)}\left(X^{(i)}(t), t\right)dt + \sum_{j=1}^{n} \sigma^{(i,j)}\left(X^{(i)}, t\right)dW^{(j)}(t) , i = 1,2, \dots, m, \tag{12}$$

where for $i, j = 1,2, \dots n$

$$dW^{(i)}(t)dW^{(j)}(t) = \begin{cases} \rho_{i,j}dt, & i \neq j, \ \rho \in (-1,1), \\ dt, & i = j. \end{cases} \tag{13}$$

Similarly, let ρ and f be given function. By denoting $X(t) = [X^{(1)}(t), \dots, X^{(m)}(t)]$, for $t \in [0, T]$ where $T > 0$ and if

$$V(X(t), t) = e^{-\int_t^T \rho(u)du} E\left[f\left(X(T)\right)\middle|X(t) = X\right], \tag{14}$$

then $V(X(t), t)$ solves the following PDE

$$\frac{dV(X(t),t)}{dx}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{n}\sum_{l=1}^{n}\sigma^{(i,k)}\left(X^{(i)}(t),t\right)\sigma^{(j,l)}\left(X^{(j)}(t),t\right)\right)\frac{d^2V(X(t),t)}{dX^{(i)}(t)dX^{(j)}(t)}$$

$$\sum_{i=1}^{m}\mu^{(i)}\left(X^{(i)}(t),t\right)\frac{dV(X(t),t)}{dX^{(i)}(t)}-\rho(t)V(X(t),t)=0, \quad (15)$$

subject to boundary condition $V(X(T),T)=f(X(T))$.

Consequently, by solving the boundary value problems (11) or (15), the expectations for solutions of SDEs can be computed. Recently, Adomian decomposition method (ADM) has been applied with a great success to obtain approximate solutions for a large variety of linear and nonlinear problems in ODEs, PDEs, and integral equations. In addition, some modifications of the ADM have been suggested [1,7-14]. The ADM approximates the solution as an infinite series and usually converges to the exact solution. In addition, the convergency of the method is considered [15].

The organization of this paper is as follows. In Section 2, for convenience of the reader, some basic definitions, and mathematical preliminaries of the stochastic calculus are presented. In Section 3, by using Feynman-Kac theorem, the PDE corresponding to Heston model is achieved. Afterwards, to solve this PDE, the ADM is applied. Finally, in Section 4, numerical example is presented.

2. BASIC CONCEPTS OF THE STOCHASTIC CALCULUS

In this section, some preliminary are reviewed in stochastic calculus [2-5]. In addition, the stochastic CIR model is solved in this section.

Definition 2.1. Brownian motion is a stochastic process $\{W(t)|t\geq 0\}$ with the following properties:

1. $W(0)=0$.

2. It has a continuous path.

3. For all non-overlapping time intervals $[t_1,t_2]$, and $[t_3,t_4]$ the random variables $W(t_2)-W(t_1)$ and $W(t_4)-W(t_3)$ are independent.

4. For all s and t, $t>s$, the increment $W(t)-W(s)$ is a normal variable, with zero mean and variance $t-s$ i.e., $W(t)-W(s)\sim N(0,t-s)$.

Definition 2.2. A process $X(t)$ is called F_t-adapted, if for all t, $X(t)$ is F_t-measurable.

Definition 2.3. An Ito process has the form

$$X(t)=X(0)+\int_0^t f(s)ds+\int_0^t g(s)dW(s), 0\leq t\leq T, \quad (16)$$

where $f(t)$ and $g(t)$ are F_t-adapted, such that $\int_0^T |f(t)|dt<\infty$ and $\int_0^T |g(t)|^2dt<\infty$. It is said that the process $X(t)$ has the stochastic differential on $[0,T]$, and

$$dX(t)=f(t)dt+g(t)dW(t), 0\leq t\leq T. \quad (17)$$

Let $X(t)$ and $Y(t)$ are Ito processes. Then the following properties are satisfied.

1. $\forall\alpha,\beta\in R$,

$$\int_0^T(\alpha X(t)+\beta Y(t))dt=\alpha\int_0^T X(t)dW(t)+\beta\int_0^T Y(t)dW(t). \quad (18)$$

2. Let $0\leq a\leq b\leq T$, then

$$\int_0^T X(t)I_{(a,b]}dt=\int_a^b X(t)dt. \quad (19)$$

3. **Zero mean property**

$$E\left[\int_0^T X(t)dW(t)\right] = 0. \tag{20}$$

In other words, the expectation of an Ito integral is zero. Furthermore, if $\int_0^T E[X^2(t)]dt < \infty$, we have

4. **Isometry property**

$$E\left[\left(\int_0^T X(t)dW(t)\right)^2\right] = \int_0^T E[X^2(t)]dt. \tag{21}$$

In the following, the second equation in (8) or the volatility model is solved.

Theorem 2.1. Let

$$\begin{cases} d\sigma^2(t) = \kappa\big(\theta - \sigma^2(t)\big)dt + \alpha\sigma(t)dW^\sigma(t), \\ \qquad\qquad \sigma(0) = \sigma_0. \end{cases} \tag{22}$$

Then the exact solution is given by

$$\sigma^2(t) = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \alpha e^{-\kappa t}\int_0^t e^{\kappa s}\sigma(s)dW^\sigma(s). \tag{23}$$

Proof. The equation (22) changes to the following form

$$d\sigma^2(t) + \kappa\sigma^2(t)dt = \kappa\theta dt + \alpha\sigma(t)dW^\sigma(t). \tag{24}$$

Multiplying both sides of the relation (24) by $e^{\kappa t}$ results in

$$d\big(e^{\kappa t}\sigma^2(t)\big) = \kappa\theta e^{\kappa t}dt + \alpha e^{\kappa t}\sigma(t)dW^\sigma(t). \tag{25}$$

Now, integrating both sides of the relation (25) on $[0, t]$, results in (23).

According to the above Theorem, the volatility in the Heston model has no general explicit solution. However, its mean and variance can be calculated explicitly. In the following, the mean and the variance of $\sigma^2(t)$ in the Heston model are computed.

Theorem 2.2. The expectation and variance of $\sigma^2(t)$ are given by

$$\begin{cases} E[\sigma^2(t)] = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}), \\ Var[\sigma^2(t)] = \frac{\alpha^2}{\kappa}\sigma_0(e^{-\kappa t} - e^{-2\kappa t}) + \frac{\theta\alpha^2}{2\kappa}(1 - e^{-\kappa t})^2. \end{cases} \tag{26}$$

Proof. Taking the expectation from both sides of the relation (23) result in

$$E[\sigma^2(t)] = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + E[\alpha e^{-\kappa t}\int_0^t e^{\kappa s}\sigma(s)dW^\sigma(s)]$$

$$= e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \quad \alpha e^{-\kappa t}\int_0^t e^{\kappa s}\sigma(s)E[dW^\sigma(s)]$$

$$= e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}). \tag{27}$$

It is interesting to note that $\lim_{t\to\infty} E[\sigma^2(t)] = \theta$. Finally, the variance can be calculated as

$$Var[\sigma^2(t)] = E[\sigma^4(t)] - (E[\sigma^2(t)])^2$$

$$= E\left[\left(e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \alpha e^{-\kappa t}\int_0^t e^{\kappa s}\sigma(s)dW^\sigma(s)\right)^2\right] - [\,e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t})]^2$$

$$= \frac{\alpha^2}{\kappa}\sigma_0(e^{-\kappa t} - e^{-2\kappa t}) + \frac{\theta\alpha^2}{2\kappa}(1 - e^{-\kappa t})^2 \tag{28}$$

The above relation is simplified by using (20) and (21). In the next section, the ADM is described. Furthermore, this method is applied to solve the PDE achieved by Feynman-Kac formula for Heston model.

3. THE ADOMIAN DECOMPOSITION METHOD (ADM)

Consider the following differential equation

$$Lu + Ru + Nu = g, \tag{29}$$

where L is the highest order derivative which assumed to be easily invertible, R is a linear differential operator of less order than L, N represents the nonlinear terms, and g is source term. Applying L^{-1} to both side of the relation (29) and using initial conditions results in

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{30}$$

where the function f represents terms arising from integrating the source term g and from using the given conditions. Let $u = \sum_{i=1}^{\infty} u_i$. In this method, the components $u_0, u_1, u_2, \ldots$ are determined recursively as follows [1,7-14]

$$\begin{cases} u_0 = f, \\ u_k = -L^{-1}(Ru_{k-1}) - L^{-1}(Nu_{k-1}), k \in N. \end{cases} \tag{31}$$

The series have been obtained is convergent when the ration of $||u_i||_\infty$ to $||u_{i-1}||_\infty$ for $i = 1,2,3, \ldots$ decrease to zero [15].

In the following, by using the Feynman-Kac formula, the ADM is applied to obtain an explicit formula for the moments of the Heston model. By using Feynman-Kac formula, to compute the n-moment of the $S(t)$ or $\sigma^2(t)$ in Heston model, the following PDE with boundary condition $V(x, y, 0) = x^n$ or $V(x, y, 0) = y^n$ is achieved

$$-\frac{dV}{dt} + \mu x \frac{dV}{dx} + \kappa(\theta - y)\frac{dV}{dy} + \frac{1}{2}x^2 y \frac{d^2V}{dx^2} + \frac{1}{2}\alpha^2 y \frac{d^2V}{dy^2} + \rho\alpha x y \frac{d^2V}{dydx} = 0. \tag{32}$$

Integrating both sides of the relation (32) results in

$$-\int_t \frac{dV}{ds} ds + \int_t \mu x \frac{dV}{dx} ds + \int_t \kappa(\theta - y)\frac{dV}{dy} ds +$$

$$\int_t \frac{1}{2}x^2 y \frac{d^2V}{dx^2} ds + \int_t \frac{1}{2}\alpha^2 y \frac{d^2V}{dy^2} ds + \int_t \rho\alpha x y \frac{d^2V}{dydx} ds = 0. \tag{33}$$

Therefore, by using initial condition $V(x, y, 0) = x^n$, the relation (33) is simplified as

$$V = x^n + \int_t \mu x \frac{dV}{dx} ds + \int_t \kappa(\theta - y)\frac{dV}{dy} ds + \int_t \frac{1}{2}x^2 y \frac{d^2V}{dx^2} ds$$

$$+ \int_t \frac{1}{2}\alpha^2 y \frac{d^2V}{dy^2} ds + \int_t \rho\alpha x y \frac{d^2V}{dydx} ds = 0. \tag{34}$$

Let $V = \sum_{n=0}^{\infty} V_n$ and $L = \frac{d}{dt}$. According to the relation (31), the components $V_0, V_1, V_2, \ldots$ are determined as follows

$$\begin{cases} V_{0=x^n,} \\ V_k = \int_t \mu x \frac{dV_{k-1}}{dx} ds + \int_t \kappa(\theta - y)\frac{dV_{k-1}}{dy} ds + \int_t \frac{1}{2}x^2 y \frac{d^2V_{k-1}}{dx^2} ds \\ \qquad + \int_t \frac{1}{2}\alpha^2 y \frac{d^2V_{k-1}}{dy^2} + \int_t \rho\alpha x y \frac{d^2V_{k-1}}{dydx}, k \in N. \end{cases} \tag{35}$$

Similarly, by using initial condition $V(x, y, 0) = y^n$, the moments of $\sigma^2(T)$ can be calculated as well.

4. Numerical example

In order to illustrate the method, the following example is considered. A program code in the Maple for $\mu = 0.03$, $\kappa = 9.9$, $\theta = 0.04$, $\alpha = 0.14$, and $\rho = -0.9$ can be done by the following procedure:

```
restart;
N:=5:
mu:=0.03:
kappa:=9.9:
theta:=0.04:
alpha:=0.14:
rho:=-0.9:
v[0]:=x**2:
for i from 1 to N do
v[i]:=simplify(int(mu*x*diff(v[i-1],x),s)+int(kappa*(theta-y)*diff(v[i-1],y),s)+int((1/2)*x**2*y*diff(v[i-1],x,x),s)+int((1/2)*(alpha**2)*y*diff(v[i-1],y,y),s)+int(rho*alpha*x*y*diff(v[i-1],x,y),s));
od;
approximate:=add(v[i](x),i=0..N);
```

Thus, the first few components, are obtained as follows

$V_0 = x^2,$

$V_1 = 0.06x^2 s + yx^2 s,$

$V_2 = 0.1998x^2 s^2 - 5.016yx^2 s^2 + 0.5y^2 x^2 s^2,$

$V_3 = -0.658116x^2 s^3 + 17.07569067yx^2 s^3 - 5.046y^2 x^2 s^3 + 0.1666666666y^3 x^2 s^3,$

$V_4 = 1.680621636x^2 s^4 - 44.27032996yx^2 s^4 + 29.85867867y^2 x^2 s^4 - 2.528y^3 x^2 s^4$
$+0.04166666665y^4 x^2 s^4,$

$V_5 = -3.486042673x^2 s^5 + 94.53801904yx^2 s^5 - 130.3762663y^2 x^2 s^5 + 21.35413334y^3 x^2 s^5$
$-0.8434999999y^4 x^2 s^5 + 0.008333333333y^5 x^2 s^5,$

$V_6 = 6.204648831x^2 s^6 - 177.2295181yx^2 s^6 + 460.0832479y^2 x^2 s^6 - 130.1486338y^3 x^2 s^6$
$+9.262417446y^4 x^2 s^6 - 0.211y^5 x^2 s^6 + 0.001388888888y^6 x^2 s^6,$

.
.
.

Finally, the series have been obtained is convergent because the ratio of $||V_i||_\infty$ to $||V_{i-1}||_\infty$ for $i = 1,2,3, ...$ slowly decrease to zero [15]. Below, these ratios for the first few are expressed

$\frac{||V_1||_\infty}{||V_0||_\infty} = 1,$

$\frac{||V_2||_\infty}{||V_1||_\infty} = 5.016,$

$\frac{||V_3||_\infty}{||V_2||_\infty} = 3.404244551,$

$\frac{||V_4||_\infty}{||V_3||_\infty} = 2.592593812,$

$\frac{||V_5||_\infty}{||V_4||_\infty} = 2.945003266,$

$\frac{||V_6||_\infty}{||V_5||_\infty} = 3.528888048,$

$\frac{||V_7||_\infty}{||V_6||_\infty} = 2.997415342,$

.
.
.

5. Conclusions

By using the Feynman-Kac formula a PDE for computing the moments of a solution for SDE is obtained. In this paper, the PDE related to Heston model is obtained. Afterwards, the ADM is applied to solve this PDE. Numerical example shows the simplicity and efficiency of the proposed method.

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