Tamap Journal of Mathematics and Statistics
http://www.tamap.org/

doi:10.29371/2018.16.50
Volume 2018, Article ID 50
Research Article

On Distributions of Order Statistics of Discrete Vectors

Mehmet GÜNGÖR [1,*] and Yunus BULUT [2]
[1,2] Department of Econometrics, University of Inonu, 44280 Malatya, Turkey

Received: 15.10.2018 • **Accepted**: 23.11.2018 • **Published Online:** 03.12.2018

Abstract: In this study, joint distributions of order statistics of nonidentically distributed discrete random vectors are expressed in form of integral by permanent. Then, results related to *pf* and *df* of order statistics of the discrete random vectors are given.

***Keywords**:* Order statistics, discrete random vector, probability function, distribution function, permanent.

1. INTRODUCTION

Several identities and recurrence relations for probability density function(*pdf*) and distribution function(*df*) of order statistics of independent and identically distributed(*iid*) random variables were established by numerous authors including Arnold et al.[1], Balasubramanian and Beg[2], David[3], and Reiss[4]. Furthermore, Arnold et al.[1], David[3], Gan and Bain[5], and Khatri[6] obtained the probability function(*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan[7] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. Nagaraja[8] explored the behavior of higher order conditional probabilities of order statistics in a attempt to understand the structure of discrete order statistics. Nagaraja[9] considered some results on order statistics of a random sample taken from a discrete population. Corley[10] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller[11]. Guilbaud[12] expressed the probability of the functions of independent but not necessarily identically distributed(*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West[13]. In addition, Vaughan and Venables[14] derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan[15], and Bapat and Beg[16] obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n)$ was obtained by Childs and Balakrishnan[17] by adding another independent random variable to the original n variables $X_1, X_2, \dots, X_n$. Also, Balasubramanian et al.[18] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg[19] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al.[20] derived the expressions for the distribution and density functions by Ryser's method and the distributions of maxima and minima based on permanents. In the first of two papers, Balasubramanian et al.[21] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, \dots, X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al.[22] generalized their previous results[21] to the case of the joint distribution function of several order statistics.

In this study, joint distributions of p order statistics of *innid* discrete random vectors are expressed in form of an integral. As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random vectors.

* Correspondence: mgungor44@gmail.com

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1, \mathrm{a}_2, \ldots$ are defined as column vectors, then matrix obtained by taking m_1 copies of a_1, m_2 copies of a_2, ... can be denoted $[\underset{m_1}{\mathrm{a}_1}\ \underset{m_2}{\mathrm{a}_2}\ \ldots]$ and perA denotes permanent of a square matrix A, which is defined as similar to determinant except that all terms in expansion have a positive sign.

Consider $\mathrm{x} = \left(x^{(1)}, x^{(2)}, \ldots, x^{(b)}\right)$ and $\mathrm{y} = \left(y^{(1)}, y^{(2)}, \ldots, y^{(b)}\right)$ $(b = 1,2,\ldots,n)$ are vector, then it can be written as $\mathrm{x} \le \mathrm{y}$ if $x^{(s)} \le y^{(s)}$ $(s = 1,2,\ldots,b)$ and $\mathrm{x} \mp \mathrm{y} = \left(x^{(1)} \mp y^{(1)}, x^{(2)} \mp y^{(2)}, \ldots, x^{(b)} \mp y^{(b)}\right)$.

Let $\xi_i = \left(\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(b)}\right)$ $(i = 1,2,\ldots,n)$ be n *innid* discrete random vectors which components of ξ_i are independent.

$$X_{r:n}^{(s)} = Z_{r:n}\left(\xi_1^{(s)}, \xi_2^{(s)}, \ldots, \xi_n^{(s)}\right) \tag{1}$$

is stated as rth order statistic of sth components of $\xi_1, \xi_2, \ldots, \xi_n$. From (1), ordered values of sth components of $\xi_1, \xi_2, \ldots, \xi_n$ are expressed as

$$X_{1:n}^{(s)} \le X_{2:n}^{(s)} \le \ldots \le X_{n:n}^{(s)}. \tag{2}$$

From (2), we can write

$$\mathrm{X}_{r:n} = \left(X_{r:n}^{(1)}, X_{r:n}^{(2)}, \ldots, X_{r:n}^{(b)}\right) \quad (1 \le r \le n).$$

Also, $\mathrm{x}_w = \left(x_w^{(1)}, x_w^{(2)}, \ldots, x_w^{(b)}\right)$, $\left(x_w^{(s)} = 0, 1, 2, \ldots\right)(w = 1,2,\ldots,p;\ p = 1,2,\ldots,n)$ and $x_0^{(s)} = 0$. Let f_i and F_i be *pf* and *df* of $\xi_i^{(s)}$, respectively.

In this study, *pf* and *df* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, \ldots, \mathrm{X}_{r_p:n}$ $\left(1 \le r_1 < r_2 < \ldots < r_p \le n\right)$ are given. Let $\mathrm{X}^{(s)} = \left(X_{r_1:n}^{(s)}, X_{r_2:n}^{(s)}, \ldots, X_{r_p:n}^{(s)}\right)$ and $\mathrm{x}^{(s)} = \left(x_1^{(s)}, x_2^{(s)}, \ldots, x_p^{(s)}\right)$. For notational convenience we write $\sum\limits_{z_1^{(s)}, z_2^{(s)}, \ldots, z_p^{(s)}}$,

$\sum\limits_{m_p, k_p, \ldots, m_1, k_1}$, $\int$ and $\int\limits_{\mathrm{V}}$ instead of $\sum\limits_{z_1^{(s)}=0}^{x_1^{(s)}} \sum\limits_{z_2^{(s)}=z_1^{(s)}}^{x_2^{(s)}} \sum\limits_{z_3^{(s)}=z_2^{(s)}}^{x_3^{(s)}} \cdots \sum\limits_{z_p^{(s)}=z_{p-1}^{(s)}}^{x_p^{(s)}}$,

$\sum\limits_{m_p=0}^{n-r_p} \sum\limits_{k_p=0}^{r_p-r_{p-1}-1} \cdots \sum\limits_{m_2=0}^{r_3-r_2-1} \sum\limits_{k_2=0}^{r_2-r_1-1} \sum\limits_{m_1=0}^{r_2-r_1-1} \sum\limits_{k_1=0}^{r_1-1}$, $\int\limits_{F_{\varsigma_2^{(1)}}(x_1^{(s)}-)}^{F_{\varsigma_2^{(1)}}(x_1^{(s)})} \int\limits_{F_{\varsigma_4^{(1)}}(x_2^{(s)}-)}^{F_{\varsigma_4^{(1)}}(x_2^{(s)})} \cdots \int\limits_{F_{\varsigma_{2p}^{(1)}}(x_p^{(s)}-)}^{F_{\varsigma_{2p}^{(1)}}(x_p^{(s)})}$ and

$\int\limits_{0}^{F_{\varsigma_2^{(1)}}(x_1^{(s)})} \int\limits_{v_{\varsigma_2^{(1)}}^{(s,1)}}^{F_{\varsigma_4^{(1)}}(x_2^{(s)})} \cdots \int\limits_{v_{\varsigma_{2(p-1)}^{(1)}}^{(s,p-1)}}^{F_{\varsigma_{2p}^{(1)}}(x_p^{(s)})}$ in the expressions below, respectively.

2. THEOREMS FOR PROBABILITY AND DISTRIBUTION FUNCTIONS

In this section, theorems related to *pf* and *df* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, \ldots, \mathrm{X}_{r_p:n}$ are given. We now express the following theorem for joint *pf* of order statistics of *innid* discrete random vectors.

Theorem 2.1.

$$f_{r_1,r_2,\ldots,r_p:n}\left(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_p\right) = \prod_{s=1}^{b} \mathrm{D} \sum_{n_{\varsigma_1}, n_{\varsigma_2}, \ldots, n_{\varsigma_{2p}}} \int \left[\prod_{w=1}^{p+1} per[\underset{r_w-r_{w-1}-1}{\mathrm{v}^{(s,w)} - \mathrm{v}^{(s,w-1)}}][\varsigma_{2w-1}/\cdot)\right] \prod_{w=1}^{p} per[\underset{1}{\mathrm{dv}^{(s,w)}}][\varsigma_{2w}/\cdot), \tag{3}$$

$\mathrm{x}_1 < \mathrm{x}_2 < \cdots < \mathrm{x}_p$, where $\mathrm{D} = \prod_{w=1}^{p+1}[(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$, $r_{p+1} = n + 1$,

$v_{\varsigma_{2w-1}^{(j)}}^{(s,t)} = [v_{\varsigma_{2t}^{(1)}}^{(s,t)} - F_{\varsigma_{2t}^{(1)}}(x_t^{(s)}-)] \frac{f_{\varsigma_{2w-1}^{(j)}}(x_w^{(s)})}{f_{\varsigma_{2t}^{(1)}}(x_t^{(s)})} + F_{\varsigma_{2w-1}^{(j)}}(x_t^{(s)}-)$, $\mathrm{v}^{(s,w)} = (v_1^{(s,w)}, v_2^{(s,w)}, \ldots, v_n^{(s,w)})'$, $\mathrm{dv}^{(s,w)} = (dv_1^{(s,w)}, dv_2^{(s,w)}, \ldots, dv_n^{(s,w)})'$, $\mathrm{v}^{(s,0)} = 0 = (0,0,\ldots,0)'$ and $\mathrm{v}^{(s,p+1)} = 1 = (1,1,\ldots,1)'$ are column vectors,

$\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}$ denotes sum over $\bigcup_{\ell=1}^{2p}\varsigma_\ell$ for which $\varsigma_\upsilon\cap\varsigma_\vartheta=\phi$ for $\upsilon\neq\vartheta$, $\bigcup_{\ell=1}^{2p+1}\varsigma_\ell=\{1,2,\dots,n\}$ and

$$\varsigma_\ell=\begin{cases}\left\{\varsigma_\ell^{(1)}\right\}, & \text{if } \ell \text{ even}\\ \left\{\varsigma_\ell^{(1)},\varsigma_\ell^{(2)},\dots,\varsigma_\ell^{\left(r_{\frac{\ell+1}{2}}-r_{\frac{\ell-1}{2}}-1\right)}\right\}, & \text{if } \ell \text{ odd.}\end{cases}$$

Here, n_{ς_ℓ} is cardinality of ς_ℓ. $\mathrm{A}[\varsigma_\ell/\cdot)$ is matrix obtained from A by taking rows whose indices are in ς_ℓ.

Proof. It can be written

$$\begin{aligned}f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)&=P\left\{\mathrm{X}_{r_1:n}=\mathrm{x}_1,\mathrm{X}_{r_2:n}=\mathrm{x}_2,\dots,\mathrm{X}_{r_p:n}=\mathrm{x}_p\right\}\\&=P\left\{\mathrm{X}^{(1)}=\mathrm{x}^{(1)},\mathrm{X}^{(2)}=\mathrm{x}^{(2)},\dots,\mathrm{X}^{(b)}=\mathrm{x}^{(b)}\right\}\\&=\prod_{s=1}^{b}P\left\{\mathrm{X}^{(s)}=\mathrm{x}^{(s)}\right\}\\&=\prod_{s=1}^{b}P\left\{X_{r_1:n}^{(s)}=x_1^{(s)},X_{r_2:n}^{(s)}=x_2^{(s)},\dots,X_{r_p:n}^{(s)}=x_p^{(s)}\right\}\\&=\prod_{s=1}^{b}f_{r_1,r_2,\dots,r_p:n}\left(x_1^{(s)},x_2^{(s)},\dots,x_p^{(s)}\right).\end{aligned}$$

Consider

$$\left\{X_{r_1:n}^{(s)}=x_1^{(s)},X_{r_2:n}^{(s)}=x_2^{(s)},\dots,X_{r_p:n}^{(s)}=x_p^{(s)}\right\}.$$

The above event can be realized mutually exclusive as follows: r_1-1-k_1 observations are less than $x_1^{(s)}$, k_w+1+m_w $(w=1,2,\dots,p)$ observations are equal to $x_w^{(s)}$, $r_\xi-1-k_\xi-m_{\xi-1}-r_{\xi-1}$ $(\xi=2,3,\dots,p)$ observations are in interval $\left(x_{\xi-1}^{(s)},x_\xi^{(s)}\right)$ and $n-m_p-r_p$ observations exceed $x_p^{(s)}$. Probability function of the above event can be written as

$$f_{r_1,r_2,\dots,r_p:n}\left(x_1^{(s)},x_2^{(s)},\dots,x_p^{(s)}\right)=P\left\{X_{r_1:n}^{(s)}=x_1^{(s)},X_{r_2:n}^{(s)}=x_2^{(s)},\dots,X_{r_p:n}^{(s)}=x_p^{(s)}\right\}.$$

The following expression can be written from the last identity.

$$\begin{aligned}&f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)\\&=\prod_{s=1}^{b}\sum_{m_p,k_p,\dots,m_1,k_1}\mathrm{C}_1 per[\underset{r_1-1-k_1}{\mathrm{F}(x_1^{(s)}-)}\;\underset{k_1+1+m_1}{\mathrm{f}(x_1^{(s)})}\;\underset{r_2-1-k_2-m_1-r_1}{\mathrm{F}(x_2^{(s)}-)-\mathrm{F}(x_1^{(s)})}\;\underset{k_2+1+m_2}{\mathrm{f}(x_2^{(s)})}\;\cdots\;\underset{k_p+1+m_p}{\mathrm{f}(x_p^{(s)})}\;\underset{n-m_p-r_p}{1-\mathrm{F}(x_p^{(s)})}],\end{aligned}$$

where $\mathrm{C}_1=\left(\prod_{w=1}^{p+1}[(r_w-1-k_w-m_{w-1}-r_{w-1})!]^{-1}\right).\prod_{w=1}^{p}[(k_w+1+m_w)!]^{-1}$, $m_0=0$, $k_{p+1}=0$,

$m_{w-1}+k_w\leq r_w-r_{w-1}-1$, $\mathrm{F}(x_w^{(s)})=\left(F_1(x_w^{(s)}),F_2(x_w^{(s)}),\dots,F_n(x_w^{(s)})\right)'$,

$\mathrm{f}(x_w^{(s)})=\left(f_1(x_w^{(s)}),f_2(x_w^{(s)}),\dots,f_n(x_w^{(s)})\right)'$ and $F_i(x_w^{(s)}-)=P\left(X_i^{(s)}<x_w^{(s)}\right)$ $(i=1,2,\dots,n)$.

In the above identity, using properties of permanent, it can be written

$$\begin{aligned}f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)=\prod_{s=1}^{b}\sum_{m_p,k_p,\dots,m_1,k_1}\mathrm{C}_1\sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}}[\prod_{w=1}^{p+1}per[\underset{m_{w-1}}{\mathrm{f}(x_{w-1}^{(s)})}][s_{4(w-1)}/\cdot)\\ \cdot\, per[\underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{\mathrm{F}(x_w^{(s)}-)-\mathrm{F}(x_{w-1}^{(s)})}][s_{4w-3}/.)per[\underset{k_w}{\mathrm{f}(x_w^{(s)})}][s_{4w-2}/.)]\prod_{w=1}^{p}per[\underset{1}{\mathrm{f}(x_w^{(s)})}][s_{4w-1}/.),\end{aligned}\tag{4}$$

where $\sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}}$ denotes sum over $\bigcup_{l=1}^{4p}s_l$ for which $s_\upsilon\cap s_\vartheta=\phi$ for $\upsilon\neq\vartheta$, $\bigcup_{l=1}^{4p+1}s_l=\{1,2,\dots,n\}$,

$\mathrm{F}\big(x_0^{(s)}\big) = 0 = (0,0,\dots,0)'$, $\mathrm{F}\big(x_{p+1}^{(s)}\big) = 1 = (1,1,\dots,1)'$ and

$$s_l = \begin{cases} \left\{ s_l^{(1)}, s_l^{(2)}, \dots, s_l^{\left(m_{\frac{l}{4}}\right)} \right\}, & \text{if } l \equiv 0 \ (\mathrm{mod}\ 4) \\ \left\{ s_l^{(1)}, s_l^{(2)}, \dots, s_l^{\left(r_{\frac{l+3}{4}}-1-m_{\frac{l-1}{4}}-k_{\frac{l+3}{4}}-r_{\frac{l-1}{4}}\right)} \right\}, & \text{if } l \equiv 1 \ (\mathrm{mod}\ 4) \\ \left\{ s_l^{(1)}, s_l^{(2)}, \dots, s_l^{\left(k_{\frac{l+2}{4}}\right)} \right\}, & \text{if } l \equiv 2 \ (\mathrm{mod}\ 4) \\ \left\{ s_l^{(1)} \right\}, & \text{if } l \equiv 3 \ (\mathrm{mod}\ 4). \end{cases}$$

(4) can be written as

$$f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p) = \prod_{s=1}^{b} \sum_{m_p,k_p,\dots,m_1,k_1} \mathrm{C}_1 \sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}} \left[\prod_{w=1}^{p} \frac{(k_w+1+m_w)!}{k_w!\,m_w!} \right]$$

$$.\left[\int_0^1 \int_0^1 \cdots \int_0^1 \left[\prod_{w=1}^{p+1} \big[y_w^{(s)}\big]^{k_w} \big[1 - y_{w-1}^{(s)}\big]^{m_{w-1}} \right] \prod_{w=1}^{p} dy_w^{(s)} \right] \left[\prod_{w=1}^{p+1} \underset{m_{w-1}}{per}[\mathrm{f}(x_{w-1}^{(s)})]\big[s_{4(w-1)}/\cdot\big) \right.$$

$$\left. \cdot \underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{per}[\mathrm{F}(x_w^{(s)}-) - \mathrm{F}(x_{w-1}^{(s)})][s_{4w-3}/\cdot) \underset{k_w}{per}[\mathrm{f}(x_w^{(s)})][s_{4w-2}/\cdot) \right] \prod_{w=1}^{p} \underset{1}{per}[\mathrm{f}(x_w^{(s)})][s_{4w-1}/\cdot).$$

The above identity can be expressed as

$$f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p) = \prod_{s=1}^{b} \sum_{m_p,k_p,\dots,m_1,k_1} \mathrm{C} \sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}}$$

$$\int_0^1 \int_0^1 \cdots \int_0^1 \left[\prod_{w=1}^{p+1} \underset{m_{w-1}}{per}\big[\big(1 - y_{w-1}^{(s)}\big)\mathrm{f}(x_{w-1}^{(s)})\big] \big[s_{4(w-1)}/\cdot\big) \underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{per}[\mathrm{F}(x_w^{(s)}-) - \mathrm{F}(x_{w-1}^{(s)})][s_{4w-3}/\cdot) \right.$$

$$\left. \cdot \underset{k_w}{per}[y_w^{(s)}\mathrm{f}(x_w^{(s)})][s_{4w-2}/\cdot) \right] \prod_{w=1}^{p} \underset{1}{per}[dy_w^{(s)}\mathrm{f}(x_w^{(s)})][s_{4w-1}/\cdot), \qquad (5)$$

where $\mathrm{C} = \left(\prod_{w=1}^{p+1} [(r_w - 1 - k_w - m_{w-1} - r_{w-1})!]^{-1}\right). \prod_{w=1}^{p} [m_w!\,k_w!]^{-1}$.

In (5), if $\mathrm{v}^{(s,w)} = y_w^{(s)}\mathrm{f}(x_w^{(s)}) + \mathrm{F}(x_w^{(s)}-)$, the following identity is obtained.

$$f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p) = \prod_{s=1}^{b} \sum_{m_p,k_p,\dots,m_1,k_1} \mathrm{C} \sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}}$$

$$\int_{F_{s_3^{(1)}}(x_1^{(s)}-)}^{F_{s_3^{(1)}}(x_1^{(s)})} \int_{F_{s_7^{(1)}}(x_2^{(s)}-)}^{F_{s_7^{(1)}}(x_2^{(s)})} \cdots \int_{F_{s_{4p-1}^{(1)}}(x_p^{(s)}-)}^{F_{s_{4p-1}^{(1)}}(x_p^{(s)})} \left[\prod_{w=1}^{p+1} \underset{m_{w-1}}{per}[\mathrm{F}(x_{w-1}^{(s)}) - \mathrm{v}^{(s,w-1)}] \big[s_{4(w-1)}/.\big) \right.$$

$$\cdot \underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{per}[\mathrm{F}(x_w^{(s)}-)-\mathrm{F}(x_{w-1}^{(s)})][s_{4w-3}/.)\underset{k_w}{per}[\mathrm{v}^{(s,w)}-\mathrm{F}(x_w^{(s)}-)][s_{4w-2}/.)\Bigg]\prod_{w=1}^{p}\underset{1}{per}[\mathrm{dv}^{(s,w)}][s_{4w-1}/\cdot). \tag{6}$$

Considering

$$\sum_{k_w=0}^{\vartheta}\sum_{m_{w-1}=0}^{\vartheta}\sum_{n_{s_{4(w-1)}},n_{s_{4w-3}}}\frac{1}{(\vartheta-k_w-m_{w-1})!\,m_{w-1}!\,k_w!}$$

$$\cdot \underset{m_{w-1}}{per}[\mathrm{G}^{(1)}][s_{4(w-1)}/\cdot)\underset{\vartheta-k_w-m_{w-1}}{per}[\mathrm{G}^{(2)}][s_{4w-3}/\cdot)\underset{k_w}{per}[\mathrm{G}^{(3)}][s_{4w-2}/\cdot)$$

$$=\frac{1}{\vartheta!}\underset{\vartheta}{per}[\mathrm{G}^{(1)}+\mathrm{G}^{(2)}+\mathrm{G}^{(3)}], \tag{7}$$

where $k_w+m_{w-1}\le\vartheta$ and using (7) for each k_w and m_{w-1}, in (6), we get

$$f_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)=\prod_{s=1}^{b}\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}$$

$$\int\left[\prod_{w=1}^{p+1}\underset{r_w-r_{w-1}-1}{per}[\mathrm{F}(x_{w-1}^{(s)})-\mathrm{v}^{(s,w-1)}+\mathrm{F}(x_w^{(s)}-)-\mathrm{F}(x_{w-1}^{(s)})+\mathrm{v}^{(s,w)}-\mathrm{F}(x_w^{(s)}-)][\varsigma_{2w-1}/\cdot)\right]$$

$$\cdot\prod_{w=1}^{p}\underset{1}{per}[\mathrm{dv}^{(s,w)}][\varsigma_{2w}/\cdot),$$

where $\varsigma_{2w-1}=s_{4(w-1)}\cup s_{4w-3}\cup s_{4w-2}$ and $\varsigma_{2w}=s_{4w-1}$. Thus, the proof is completed.

If $x_1=x_2=\dots=x_p=x$, it should be written $\int\int\dots\int$ instead of $\int$ in (3), where $\int\int\dots\int$ is to be carried out over region: $F_{\varsigma_2^{(1)}}(x^{(s)}-)\le v_{\varsigma_2^{(1)}}^{(s,1)}\le v_{\varsigma_4^{(1)}}^{(s,2)}\le\dots\le v_{\varsigma_{2p}^{(1)}}^{(s,p)}\le F_{\varsigma_{2p}^{(1)}}(x^{(s)})$, $F_{\varsigma_2^{(1)}}(x^{(s)}-)\le v_{\varsigma_2^{(1)}}^{(s,1)}\le F_{\varsigma_2^{(1)}}(x^{(s)})$, $F_{\varsigma_4^{(1)}}(x^{(s)}-)\le v_{\varsigma_4^{(1)}}^{(s,2)}\le F_{\varsigma_4^{(1)}}(x^{(s)})$,..., $F_{\varsigma_{2p}^{(1)}}(x^{(s)}-)\le v_{\varsigma_{2p}^{(1)}}^{(s,p)}\le F_{\varsigma_{2p}^{(1)}}(x^{(s)})$.

Moreover, if $x_1\le x_2\le\dots\le x_p$, it should be written $\int\int\dots\int$ instead of $\int$ in (3), where $\int\int\dots\int$ is to be carried out over region: $v_{\varsigma_2^{(1)}}^{(s,1)}\le v_{\varsigma_4^{(1)}}^{(s,2)}\le\dots\le v_{\varsigma_{2p}^{(1)}}^{(s,p)}$, $F_{\varsigma_2^{(1)}}(x_1^{(s)}-)\le v_{\varsigma_2^{(1)}}^{(s,1)}\le F_{\varsigma_2^{(1)}}(x_1^{(s)})$, $F_{\varsigma_4^{(1)}}(x_2^{(s)}-)\le v_{\varsigma_4^{(1)}}^{(s,2)}\le F_{\varsigma_4^{(1)}}(x_2^{(s)})$,..., $F_{\varsigma_{2p}^{(1)}}(x_p^{(s)}-)\le v_{\varsigma_{2p}^{(1)}}^{(s,p)}\le F_{\varsigma_{2p}^{(1)}}(x_p^{(s)})$.

We now express the following theorem for joint *df* of order statistics of *innid* discrete random vectors.

Theorem 2.2.

$$F_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)$$

$$=\prod_{s=1}^{b}\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}\int_V\left[\prod_{w=1}^{p+1}\underset{r_w-r_{w-1}-1}{per}[\mathrm{v}^{(s,w)}-\mathrm{v}^{(s,w-1)}][\varsigma_{2w-1}/.)\right]\prod_{w=1}^{p}\underset{1}{per}[\mathrm{dv}^{(s,w)}][\varsigma_{2w}/.). \tag{8}$$

Proof. It can be written

$$F_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)=\prod_{s=1}^{b}F_{r_1,r_2,\dots,r_p:n}(x_1^{(s)},x_2^{(s)},\dots,x_p^{(s)})$$

$$=\prod_{s=1}^{b}\sum_{z_1^{(s)},z_2^{(s)},\dots,z_p^{(s)}} f_{r_1,r_2,\dots,r_p:n}\left(z_1^{(s)},z_2^{(s)},\dots,z_p^{(s)}\right).$$

The above identity can be expressed as

$$F_{r_1,r_2,\dots,r_p:n}(\mathrm{x}_1,\mathrm{x}_2,\dots,\mathrm{x}_p)=\prod_{s=1}^{b}\sum_{z_1^{(s)},z_2^{(s)},\dots,z_p^{(s)}} \mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}$$

$$\int\left[\prod_{w=1}^{p+1} per[\underbrace{\mathrm{v}^{(s,w)}-\mathrm{v}^{(s,w-1)}}_{r_w-r_{w-1}-1}][\varsigma_{2w-1}/.)\right]\prod_{w=1}^{p} per[\underbrace{\mathrm{dv}^{(s,w)}}_{1}][\varsigma_{2w}/\cdot).$$

Thus, the proof is completed.

3. RESULTS FOR PROBABILITY AND DISTRIBUTION FUNCTIONS

In this section, results related to *pf* and *df* of $\mathrm{X}_{r_1:n},\mathrm{X}_{r_2:n},\dots,\mathrm{X}_{r_p:n}$ are given. We now express the following result for *pf* of rth order statistic of *innid* discrete random vectors.

Result 3.1.

$$f_{r_1:n}\left(x_1^{(1)}\right)=\frac{1}{(r_1-1)!\,(n-r_1)!}\sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)}^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per[\underbrace{\mathrm{v}^{(1,1)}}_{r_1-1}][\varsigma_1/.)\,per[\underbrace{1-\mathrm{v}^{(1,1)}}_{n-r_1}][\varsigma_3/.)per[\underbrace{\mathrm{dv}^{(1,1)}}_{1}][\varsigma_2/.)\quad(9)$$

Proof. In (3), if $b=1$, $p=1$, (9) is obtained.

In Result 3.2-3.3, *pf 's* of minimum and maximum order statistics of *innid* discrete random vectors are given, respectively.

Result 3.2.

$$f_{1:n}\left(x_1^{(1)}\right)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)}^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per[\underbrace{1-\mathrm{v}^{(1,1)}}_{n-1}][\varsigma_3/.)\,per[\underbrace{\mathrm{dv}^{(1,1)}}_{1}][\varsigma_2/.)\qquad(10)$$

Proof. In (9), if $r_1=1$, (10) is obtained.

Specially, in (10), by taking $n=2$ and $v_{\varsigma_3^{(1)}}^{(1,1)}=[v_{\varsigma_2^{(1)}}^{(1,1)}-F_{\varsigma_2^{(1)}}(x_1^{(1)}-)]\frac{f_{\varsigma_3^{(1)}}(x_1^{(1)})}{f_{\varsigma_2^{(1)}}(x_1^{(1)})}+F_{\varsigma_3^{(1)}}(x_1^{(1)}-)$, the following identity is obtained.

$$f_{1:2}\left(x_1^{(1)}\right)=\sum_{n_{\varsigma_2}=1}\int_{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)}^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per[\underbrace{1-\mathrm{v}^{(1,1)}}_{1}][\varsigma_3/.)\,per[\underbrace{\mathrm{dv}^{(1,1)}}_{1}][\varsigma_2/.)$$

$$=\sum_{n_{\varsigma_2}=1}\int_{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)}^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)}\left(1-v_{\varsigma_3^{(1)}}^{(1,1)}\right)dv_{\varsigma_2^{(1)}}^{(1,1)}$$

$$=\sum_{n_{\varsigma_2}=1}\left\{f_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)+\frac{1}{2}f_{\varsigma_3^{(1)}}\left(x_1^{(1)}\right)F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)-\frac{1}{2}f_{\varsigma_3^{(1)}}\left(x_1^{(1)}\right)F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)-f_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)F_{\varsigma_3^{(1)}}\left(x_1^{(1)}-\right)\right\}$$

$$=f_1\left(x_1^{(1)}\right)+\frac{1}{2}f_2\left(x_1^{(1)}\right)F_1\left(x_1^{(1)}-\right)-\frac{1}{2}f_2\left(x_1^{(1)}\right)F_1\left(x_1^{(1)}\right)-f_1\left(x_1^{(1)}\right)F_2\left(x_1^{(1)}-\right)$$

$$+f_2(x_1^{(1)})+\frac{1}{2}f_1(x_1^{(1)})F_2(x_1^{(1)}-)-\frac{1}{2}f_1(x_1^{(1)})F_2(x_1^{(1)})-f_2(x_1^{(1)})F_1(x_1^{(1)}-).$$

Morever, the above identity in iid case can be expressed as

$$f_{1:2}\left(x_1^{(1)}\right)=2f(x_1^{(1)})-2f(x_1^{(1)})F(x_1^{(1)})+f^2(x_1^{(1)}).$$

This result is obtained, if $i=1, n=2$ in equation (2) in [6]. Also, the above identity for $x_1^{(1)}=1$ can be written as

$$f_{1:2}(1)=2f(1)-2f(0)f(1)-f^2(1).$$

Result 3.3.

$$f_{n:n}\left(x_1^{(1)}\right)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_1}}\int_{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}-\right)}^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per\underset{n-1}{[\mathrm{v}^{(1,1)}]}[\varsigma_1/\cdot)per\underset{1}{[\mathrm{dv}^{(1,1)}]}[\varsigma_2/\cdot) \tag{11}$$

Proof. In (9), if $r_1=n$, (11) is obtained.

In the following result, we give joint *pf* of $\mathrm{X}_{1:n},\mathrm{X}_{2:n},\ldots,\mathrm{X}_{p:n}$.

Result 3.4.

$$f_{1,2,\ldots,p:n}\left(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_p\right)=\frac{1}{(n-p)!}\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int per\underset{n-p}{[1-\mathrm{v}^{(s,p)}]}[\varsigma_{2p+1}/.)\prod_{w=1}^{p}per\underset{1}{[\mathrm{dv}^{(s,w)}]}[\varsigma_{2w}/\cdot), \tag{12}$$

$\mathrm{x}_1<\mathrm{x}_2<\ldots<\mathrm{x}_p$.

Proof. In (3), if $b=1$, $r_1=1$, $r_2=2$, ...,$r_p=p$, (12) is obtained.

We now give three results for *df* of single order statistics of *innid* discrete random vectors.

Result 3.5.

$$F_{r_1:n}\left(x_1^{(1)}\right)=\frac{1}{(r_1-1)!\,(n-r_1)!}\sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_0^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per\underset{r_1-1}{[\mathrm{v}^{(1,1)}]}[\varsigma_1/.)\,per\underset{n-r_1}{[1-\mathrm{v}^{(1,1)}]}[\varsigma_3/.)per\underset{1}{[\mathrm{dv}^{(1,1)}]}[\varsigma_2/.) \tag{13}$$

Proof. In (8), if $b=1$, $p=1$, (13) is obtained.

Result 3.6.

$$F_{1:n}\left(x_1^{(1)}\right)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_2}}\int_0^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per\underset{n-1}{[1-\mathrm{v}^{(1,1)}]}[\varsigma_3/\cdot)\,per\underset{1}{[\mathrm{dv}^{(1,1)}]}[\varsigma_2/.) \tag{14}$$

Proof. In (13), if $r_1=1$, (14) is obtained.

Result 3.7.

$$F_{n:n}\left(x_1^{(1)}\right)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_1}}\int_0^{F_{\varsigma_2^{(1)}}\left(x_1^{(1)}\right)} per\underset{n-1}{[\mathrm{v}^{(1,1)}]}[\varsigma_1/.)\,per\underset{1}{[\mathrm{dv}^{(1,1)}]}[\varsigma_2/.) \tag{15}$$

Proof. In (13), if $r_1=n$, (15) is obtained.

In the following result, we give joint *df* of $\mathrm{X}_{1:n},\mathrm{X}_{2:n},\ldots,\mathrm{X}_{p:n}$.

Result 3.8.

$$F_{1,2,\ldots,p:n}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_p)=\frac{1}{(n-p)!}\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int_V per[1-\mathrm{v}^{(1,p)}]_{n-p}[\varsigma_{2p+1}/.)\prod_{w=1}^{p} per[\mathrm{dv}^{(1,w)}]_{1}[\varsigma_{2w}/.) \tag{16}$$

Proof. In (8), if $b=1$, $r_1=1$, $r_2=2$, ...,$r_p=p$, (16) is obtained.

REFERENCES

[1] Arnold, BC., Balakrishnan, N., Nagaraja, HN. (1992). “A first course in order statistics”, John Wiley and Sons Inc., New York.
[2] Balasubramanian, K., Beg, MI. (2003). “On special linear identities for order statistics”, Statistics 37, pp. 335-339.
[3] David, HA. (1981). “Order statistics”, John Wiley and Sons Inc., New York, 1981.
[4] Reiss, R. -D. (1989). “Approximate distributions of order statistics”, Springer-Verlag, New York.
[5] Gan, G., Bain, LJ. (1995). “Distribution of order statistics for discrete parents with applications to censored sampling”, J. Statist. Plann. Inference 44, pp. 37-46.
[6] Khatri, CG. (1962). “Distributions of order statistics for discrete case”, Ann. Ins. Statist. Math. 14, pp. 167-171.
[7] Balakrishnan, N. (1986). “Order statistics from discrete distributions”, Commun. Statist. Theory Meth. 15, pp. 657-675.
[8] Nagaraja, HN. (1986). “Structure of discrete order statistics”, J. Statist. Plann. Inference 13, pp. 165-177.
[9] Nagaraja, HN. (1992). “Order statistics from discrete distributions”, Statistics 23, pp. 189-216.
[10] Corley, H.W. (1984). “Multivariate order statistics”, Commun. Statist. Theory Meth. 13, pp. 1299-1304.
[11] Goldie, CM., Maller, RA. (1999). “Generalized densities of order statistics”, Statist. Neerlandica 53, pp. 222-246.
[12] Guilbaud, O. (1982). “Functions of non-i.i.d. random vectors expressed as functions of i.i.d. random vectors”, Scand. J. Statist. 9, pp. 229-233.
[13] Cao, G., West, M. (1997). “Computing distributions of order statistics”, Commun. Statist. Theory Meth. 26, pp. 755-764.
[14] Vaughan, RJ., Venables, WN. (1972). “Permanent expressions for order statistics densities”, J. Roy. Statist. Soc. Ser. B 34, pp. 308-310.
[15] Balakrishnan, N. (2007). “Permanents, order statistics, outliers and robustness”, Rev. Mat. Complut. 20, pp. 7-107.
[16] Bapat, RB., Beg, MI. (1989). “Order statistics for nonidentically distributed variables and permanents”, Sankhyā Ser. A 51, pp. 79-93.
[17] Childs, A., Balakrishnan, N. (2006). “Relations for order statistics from non-identical logistic random variables and assessment of the effect of multiple outliers on bias of linear estimators”, J. Stat. Plann. Inference 136, pp. 2227-2253.
[18] Balasubramanian, K., Balakrishnan, N., Malik, HJ. (1994). “Identities for order statistics from non-independent non- identical variables”, Sankhyā Ser. B 56, pp. 67-75.
[19] Beg, MI. (1991). “Recurrence relations and identities for product moments of order statistics corresponding to nonidentically distributed variables”, Sankhyā Ser. A 53, pp. 365-374.
[20] Cramer, E., Herle, K., Balakrishnan, N. (2009). “Permanent Expansions and Distributions of Order Statistics in the INID Case”, Commun. Statist. Theory Meth. 38, pp. 2078-2088.
[21] Balasubramanian, K., Beg, MI., Bapat, RB. (1991). “On families of distributions closed under extrema”, Sankhyā Ser. A 53, pp. 375-388.
[22] Balasubramanian, K., Beg, MI., Bapat, MI., Bapat, RB. (1996). “An identity for the joint distribution of order statistics and its applications”, J. Statist. Plann. Inference 55, pp. 13-21.