Tamap Journal of Mathematics and Statistics

Volume 2017, Article ID 11

Research Article

Operational representations of various set of polynomials using T_k operator

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Received: 12 July 2017 / Accepted: 20 August 2017

Abstract: In this paper, we analyze and study the operator representations of various polynomial sets using a new operator which was introduced by H.B. Mittal [10]. We also look forward in the literature in which M.A. Khan and A.K. Shukla [8] designed a new technique by which the finite series representation of binomial and trinomial partial differential operators can be easily grasped by the learners.

Keywords: T_k operator, Operational representation.

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1. INTRODUCTION AND PRELIMINARIES

In 1964, W.A. AL-Salam [15] defined an operator and studied the various aspect of the operator

$$\theta = x(1 + xD) \quad D \equiv \frac{d}{dx}. \tag{1.1}$$

Al-Salam used his operator in a graceful and stylish manner to derive some familiar formulae including classical orthogonal polynomials. Al-Salam also established an operator representation for the Laguerre, Jacobi, Legendre and other well - known polynomials described in the literature.

In 1971, H.B. Mittal [10] designed an operator which is generalized formof AL-Salam operator given by underneath relation

$$T_k = x(k + xD), \qquad D \equiv \frac{d}{dx}. \tag{1.2}$$

In 2010, M.A. Khan and K.S. Nisar [9] established an operator representations of Ces$\dot{a}$ro, Meixner, Sylvester, Shively's psuedo Laguerre, Jacobi, Hermite, Legendre, Gegenbauer, Ultraspherical, Bateman's Z_m(x), Bateman's generalization of $Z_m(x)$, Rice's, Sister Celine's, Bessel, Tchebycheff, Konhauser, Lagrange and Bedient polynomials by using the AL-Salam operator.

The aim of this write up is to obtain operator representations of various polynomial sets by using T_k operator [10] calculated by the technique used by M.A. Khan and A.K.Shukla [8].The result designed by us is the generalized form of the results obtained by M. A. Khan and K.S. Nisar [9].

2. The definition, notation and results used

In obtaining the operational representation of various polynomial sets by means of $T_k$operator introduced by H.B. Mittal [10]

$$T_k = x(k + xD) \ , \qquad (2.1)$$

which yields

$$T_k^m\{x^\alpha\} = (\alpha + k)_m x^{\alpha+k} \ , \qquad (2.2)$$

where k is an integer, m a non-negative integer and $\alpha$is an arbitrary.This operator is essentially that of Chak [3] and is closely related to these employed by Carlitz [4] and Gould and Hopper [7]. We find it useful in deriving operator representation of various polynomial sets.

The Leibnitz formula for the operator T is

$$T_k^m\{xuv\} = x \sum \binom{m}{k} (T_k^{m-r} v)(T_l^r u) \ , \qquad (2.3)$$

where $T_l^r = x(1 + xD)$ is the AL-Salam operator. Eq.(2.3) can be easily verified by induction. If $\frac{1}{T_k}$ is the inverse of the operator T_k, then

$$\frac{1}{T_k^m}\{x^\alpha\} = \frac{(-1)^m}{(\alpha - k + 1)_m} x^{-\alpha-m} \qquad (2.4)$$

The Pochhammer symbol is defined as

$$(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} \ . \qquad (2.5)$$

$$(\alpha)_m = \begin{cases} 1, & if\ m = 0 \\ \alpha(\alpha+1)(\alpha+2)\dots\big(\alpha + (m-1)\big), & if\ m = 1, 2, \dots, \end{cases} \qquad (2.6)$$

for (2.5), it is easy to find that

$$(\alpha)_{m-k} = \frac{(-1)^k(\alpha)_m}{(1-\alpha-m)_k}; \qquad 0 \le k \le m, \tag{2.7}$$

from [12], one obtains

$$(m-k)! = \frac{(-1)^k\, m!}{(-m)_k} \tag{2.8}$$

The hypergeometric function $F(a,b;c;z)$ has been given in [12]

$$\mathrm{F(a,b;c;z)} = \sum_{\mathrm{m=0}}^{\infty} \frac{\mathrm{(a)_m(b)_m}}{\mathrm{(c)_m}} \frac{\mathrm{z^m}}{\mathrm{m!}}, \tag{2.9}$$

it converges for $|Z| < 1$and the binomial expansion is given by

$$(1-x)^{-a} = \sum_{m=0}^{\infty} \frac{a_m x^m}{m!}; \qquad |x| < 1. \tag{2.10}$$

In particular, the following results have been used

$$T_k^r\{x^{-\alpha}\} = (-\alpha + k)_r x^{-\alpha+r} \tag{2.11}$$
$$T_k^r\{x^{-\alpha-m}\} = (-\alpha - m + k)_r x^{-\alpha-m+r} \tag{2.12}$$
$$T_k^{m-r}\{x^{-1+\alpha+m}\} = \frac{(-1+\alpha+m+k)_m(-1)^r}{(2+2m-\alpha-k)_r} x^{-1+\alpha+2m-r} \tag{2.13}$$
$$T_k^{m-r}\{x^{-\alpha}\} = \frac{(-\alpha+k)_m(-1)^r}{(1+\alpha-k-m)_r} x^{-\alpha-r+m}. \tag{2.14}$$

The definition of the following polynomials are given in terms of hypergeometric function and also their notations (see[1, 5, 12, 13, 14]).

Legendre polynomials

It is denoted by the symbol $P_m(x)$ and is defined as

$$P_m(x) = {}_2F_1\left[\begin{matrix} -m, m+1; & \frac{1-x}{2} \\ 1; & \end{matrix}\right]. \tag{2.15}$$

Hermite polynomial

It is denoted by the symbol $H_m(x)$and is defined as

$$H_m(x) = \ {}_2F_1\left[\begin{matrix}\frac{-m}{2} & \frac{-m+1}{2}; & \frac{-1}{x^2} \\ & -; & \end{matrix}\right]. \qquad (2.16)$$

Laguerre polynomial

It is denoted by $L_m^{(\alpha)}(x)$ and is defined as

$$L_m^{(\alpha)}(x) = \frac{(1+\ \alpha)_m}{m!}\ {}_2F_1\left[\begin{matrix}-m; & x \\ 1+\ \alpha; & \end{matrix}\right]. \qquad (2.17)$$

Jacobi polynomial

It is denoted by $P_m^{(\alpha,\beta)}(x)$ and is defined as

$$P_m^{(\alpha,\beta)}(x) = \frac{(1+\ \alpha)_m}{m!}\ {}_2F_1\left[\begin{matrix}-m & 1+\ \alpha+\ \beta+m\ ; & \frac{1-x}{2} \\ & 1+\ \alpha; & \end{matrix}\right]. \ (2.18)$$

Ultraspherical polynomial

The special case of Jacobi polynomial i.e., $\alpha = \beta$ is called Ultraspherical polynomial. It is denoted by $P_m^{(\alpha,\alpha)}(x)$ and is defined as

$$P_m^{(\alpha,\alpha)}(x) = \frac{(1+\ \alpha)_m}{m!}\ {}_2F_1\left[\begin{matrix}-m & 1+2\alpha+\ m\ ; & \frac{1-x}{2} \\ & 1+\ \alpha; & \end{matrix}\right]. \qquad (2.19)$$

Gegenbauer polynomial

It is the generalization of Legendre polynomial denoted by $C_m^{\nu}(x)$.It is defined as

$$C_m^{(\nu)}(x) = \frac{(2\nu)_m}{m!}\ {}_2F_1\left[\begin{matrix}-m & 2\nu+\ m\ ; & \frac{1-x}{2} \\ \nu+\frac{1}{2} & ; & \end{matrix}\right]. \ (2.20)$$

Bateman's polynomial

It is denoted by $Z_m(x)$ and is defined as

$$Z_m(x) = \ {}_2F_2\left[\begin{matrix}-m, m+1; & x \\ 1, \quad 1\ ; & \end{matrix}\right]. \qquad (2.21)$$

Rice's polynomial

It is denoted by $H_m(\xi, p, \nu)$ and is defined as

$$H_m(\xi, p, \nu) = \ {}_3F_2\left[\begin{matrix}-m, m+1, \xi; & \nu \\ 1, \quad p\ ; & \end{matrix}\right]. \qquad (2.22)$$

Cesàro polynomial

It is denoted by $g_m^{(s)}(x)$ and is defined as

$$g_m^{(s)}(x) = \binom{s\ +\ m}{m}\ {}_2F_1\left[\begin{matrix}-m, \quad 1; & x \\ -s-m\ ; & \end{matrix}\right]. \qquad (2.23)$$

Meixner polynomial

It is denoted by $M_m(x;\ \beta; c)$ and is defined as

$$M_m(x;\ \beta; c) = {}_2F_1\left[\begin{matrix}-m, -x; \\ \beta \qquad ;\end{matrix}\quad 1 - c^{-1}\right]. \qquad (2.24)$$

$$\beta > 0,\ 0 < c < 1,\ x = 0, 1, 2, \ldots.$$

Krawtochouk polynomial

It is denoted by $K_m(x; P;\ N)$and is defined as

$$K_m(x; P;\ N) = {}_2F_1\left[\begin{matrix}-m, -x; \\ N \qquad ;\end{matrix}\quad P^{-1}\right]. \qquad (2.25)$$

$$0 < P < 1,\ x = 0, 1, 2, \ldots, N$$

Hahn polynomial

It is denoted by $Q_m(x; \alpha; \beta, N)$ and is defined as

$$Q_m(x; \alpha; \beta, N) = {}_3F_2\left[\begin{matrix}-m & -x & \alpha + \beta + m + 1; \\ & -N & 1 + \alpha\end{matrix}\quad \nu\right]. \qquad (2.26)$$

$$\alpha, \beta > 0, m, x = 0, 1, 2, \ldots.$$

Slyvester polynomial

It is denoted by $\phi_m(x)$and is defined as

$$\phi_m(x) = \frac{x^m}{m!}\ {}_2F_1\left[\begin{matrix}-m & x; \\ - & ;\end{matrix}\quad x^{-1}\right]. \qquad (2.27)$$

Gottlieb polynomial

It is denoted by $l_m(x; \lambda)$and is defined as

$$l_m(x; \lambda) = e^{-m\lambda}\ {}_2F_1\left[\begin{matrix}-m & -x; \\ 1 & ;\end{matrix}\quad e^{\lambda}\right]. \qquad (2.28)$$

Charlier polynomial

It is denoted by $C_m^a(x)$and is defined as

$$C_m^a(x) = (-a)^m\ {}_2F_0\left[\begin{matrix}-m & -x; \\ - & ;\end{matrix}\quad \frac{1}{a}\right]. \qquad (2.29)$$

Mittag-Lefflerpolynomial

It is denoted by $g_m(z; \gamma)$and is defined as

$$g_m(z; \gamma) = \frac{(-\gamma)^m}{m!}\ {}_2F_1\left[\begin{matrix}-m & -z; \\ \gamma & ;\end{matrix}\quad 2\right]. \qquad (2.30)$$

Shively's pseudo Laguerrepolynomial

It is denoted by $R_m(a,x)$and is defined as

$$R_m(a,x) = \frac{(a)_{2m}}{m! \quad (a)_m} \; {}_2F_1\left[\begin{matrix} -m, & x; \\ a+m & ; \end{matrix} \quad x\right]. \tag{2.31}$$

Gegenbauer type generalized Bateman's polynomial

It is the generalization of Bateman's polynomial in the form of Gegenbauer denoted by $Z_m^v(b,x)$. It is defined as

$$Z_m^v(b,x) = {}_2F_2\left[\begin{matrix} -m & m+2v; \\ v+\frac{1}{2} & 1+b; \end{matrix} \quad x\right]. \tag{2.32}$$

Generalized Bateman's polynomial

It is denoted by $Z_m^{(\alpha,\beta)}(b,x)$ and is defined as

$$Z_m^{(\alpha,\beta)}(b,x) = {}_2F_2\left[\begin{matrix} -m, & 1+\alpha+\beta+m; \\ & 1+\alpha \quad 1+b; \end{matrix} \quad x\right]. \tag{2.33}$$

Tchebicheff polynomial of first kind $T_m(x)$

It is denoted by $T_m(x)$ and is defined as

$$T_m(x) = \frac{m!}{\left(\frac{1}{2}\right)_m} P_m^{\left(\frac{-1}{2},\frac{1}{2}\right)} \; {}_2F_1\left[\begin{matrix} -m & m; \\ \frac{1}{2}; & \end{matrix} \quad \frac{1-x}{2}\right]. \tag{2.34}$$

Tchebicheff polynomial of the second kind $U_m(x)$

It is denoted by $U_m(x)$and is defined as

$$U_m(x) = \frac{(m+1)!}{\left(\frac{3}{2}\right)_m} P_m^{\left(\frac{1}{2},\frac{1}{2}\right)} = (m+1) \; {}_2F_1\left[\begin{matrix} -m & m+2; \\ \frac{3}{2}; & \end{matrix} \quad \frac{1-x}{2}\right]. \tag{2.35}$$

Generalized Rice's polynomial$H_m^{(\alpha,\beta)}(\xi,p,\nu)$

Khandekar [11] defined a Jacobi type generalization of Rice's polynomial $H_m^{(\alpha,\beta)}(\xi,p,\nu)$. It is denoted by $H_m^{(\alpha,\beta)}(\xi,p,\nu)$ and is defined as

$$H_m^{(\alpha,\beta)}(\xi,p,\nu) = {}_3F_2\left[\begin{matrix} -m, & 1+\alpha+\beta+m, \; \xi; \\ & 1+\alpha \quad p; \end{matrix} \quad \nu\right]. \tag{2.36}$$

Bessel polynomial$y_m(x)$

It is denoted by $y_m(x)$and is defined as

$$y_m(x) = {}_2F_0\left[\begin{matrix} -m & m+1; \\ - & ; \end{matrix} \quad \frac{-x}{2}\right]. \tag{2.37}$$

Generalized Bessel polynomial $y_m(a,b,x)$

It is denoted by $y_m(a,b,x)$ and is defined as

$$y_m(a,b,x) = {}_2F_0\left[\begin{matrix}-m & a-1+m; \\ - & ;\end{matrix}\ \frac{-x}{b}\right]. \qquad (2.38)$$

Lagrange polynomial

It is denoted by $g_m^{(\alpha,\beta)}(x,y)$ and is defined as

$$g_m^{(\alpha,\beta)}(x,y) = \frac{(\alpha)_m}{m!}\ {}_2F_1\left[\begin{matrix}-m, & \beta; \\ 1-\alpha-m; & \end{matrix}\ \frac{y}{x}\right]. \qquad (2.39)$$

Sister-Celine's polynomial

Sister Celine's (Fasenmyer [6]) denoted her polynomial by the symbol

$$f_m\left[\begin{matrix}a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q\end{matrix}\ x\right].$$

It is defined as

$$f_m\left[\begin{matrix}a_1, a_2, \ldots a_p; \\ b_1 b_2 \ldots b_q\end{matrix}\ x\right] = {}_{p+2}F_{q+2}\left[\begin{matrix}-m, m+1, a_1, a_2, \ldots, a_p; \\ 1, \frac{1}{2}, \quad b_1 b_2, \ldots, b_q;\end{matrix}\ x\right]. \qquad (2.40)$$

Konhauserbi-orthogonal polynomial

It is denoted by $Z_m^{(\alpha)}(x;k)$ and is defined as

$$Z_m^{(\alpha)}(x;k) = \lim_{|\beta|\to \inf}\left\{J_m^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}. \qquad (2.41)$$

Bedient'spolynomial

Bedient [12] introduced Appell's F_2, F_3 in his study. It is defined as

$$R_m(\beta,\gamma;x) = \frac{(2x)^m(\beta)_m}{m!}\ {}_2F_1\left[\begin{matrix}-\frac{m}{2}, \frac{m}{2}+1, \gamma-\beta & ; \\ \gamma, \quad 1-\beta-m & ;\end{matrix}\ \frac{1}{x^2}\right]. \qquad (2.42)$$

and

$$G_m(\alpha,\beta;x) = \frac{(\alpha)_m(\beta)_m(2x)^m}{m!(\alpha+\beta)_m}\ {}_2F_1\left[\begin{matrix}-\frac{m}{2}, \frac{m}{2}+\frac{1}{2}, 1-\alpha-\beta-m & ; \\ 1-\alpha-m, \quad 1-\beta-m & ;\end{matrix}\ \frac{1}{x^2}\right]. \qquad (2.43)$$

3. Operator Representations

By using the technique developed by M.A.Khan and A.K.Shukla [8] the following operator representations of various polynomial sets have been obtained.

If $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$ and $D_z = \frac{\partial}{\partial z}$, we consider

$$T_k(x) = x\left(k + x\frac{\partial}{\partial x}\right)$$

$$T_l(y) = y\left(k + y\frac{\partial}{\partial y}\right)$$

and

$$T_n(z) = z\left(k + z\frac{\partial}{\partial z}\right)$$

then the binomial expansion for $(T_k+T_l)^m$ as

$$(T_k+T_l)^m = \sum_{r=0}^{\infty}\binom{m}{r} T_k^{m-r}T_l^r. \qquad (3.1)$$

which is similar to the operator $(D_x+D_y)^m$ given by M.A. Khan and A.K. Shukla [8].

$$(D_x+D_y)^m = \sum_{r=0}^{m}\binom{m}{r} D_x^{m-r}D_y^r.$$

where $\binom{m}{r} = \frac{m!}{r!\ (m-r)!}$ and also writing the finite series on the right of (3.1) as

$$(T_k+T_l)^m = \sum_{r=0}^{\infty}\binom{m}{r} T_k^r\ T_l^{m-r}, \qquad (3.2)$$

if $F(x,y)$ is a function of x and y then obtained the following from (3.1) and (3.2).

$$(T_k+T_l)^m\ F(x,y) = \sum_{r=0}^{m}\frac{(-m)_r(-1)^r}{r!} T_k^{m-r}\ T_l^r\ F(x,y) \qquad (3.3)$$

$$(T_k+T_l)^m\ F(x,y) = \sum_{r=0}^{m}\frac{(-m)_r(-1)^r}{r!} T_k^r\ T_l^{m-r}\ F(x,y) \qquad (3.4)$$

Considerably, if $F(x,y) = f(x)g(y)$ then we get the following results from (3.3) and (3.4)

$$(T_k+T_l)^m\ f(x)g(y) = \sum_{r=0}^{m}\frac{(-m)_r(-1)^r}{r!} T_k^r\ \ f(x)T_l^r\, g(y) \qquad (3.5)$$

$$= \sum_{r=0}^{m}\frac{(-m)_r(-1)^r}{r!} T_k^r\ f(x)\ T_l^{m-r}\ g(y). \qquad (3.6)$$

4. Main results

Now, by considering the particular values of $f(x)$ and $g(x)$ in (3.5)and(3.6), we obtain the following partial binomial differential operator representations of the polynomials given above

$$(T_k+T_l)^m\{x^{-m-k}y^{m+1-l}\} = (-1)^m\ m!\ x^{-k}y^{m+1-l}P_m\left(1-\tfrac{2y}{x}\right). \qquad (4.1)$$

$$(1+T_k)^m\{x^{-(m+k)+r}\} = x^{-m-k}H_m\left(\tfrac{1}{2x}\right). \qquad (4.2)$$

$$(T_k+1)^m\{x^{-\alpha-k-m}\} = (-1)^m\ m!\ \ x^{-\alpha-k}L_n^{(\alpha)}\left(\tfrac{1}{x}\right), \qquad (4.3)$$

or

$$\left(1+\frac{1}{T^k}\right)^m \{x^{-\alpha-k}\,x^{m+1-l}\} = \frac{m!}{x^{\alpha+k}(1+\alpha)_n}L_m^{(\alpha)}\left(\frac{1}{x}\right). \qquad (4.4)$$

$$(T_k+T_l)^m\left\{x^{-\alpha-m-k}y^{\alpha+\beta+m+1-l}\right\}$$
$$= (-1)^m\ m!\ x^{-\alpha-k}y^{\alpha+\beta+m+1-l}P_m^{(\alpha,\beta)}\left(1-\tfrac{2y}{x}\right). \qquad (4.5)$$

$$(T_k+T_l)^m\left\{x^{-\alpha-m-k}y^{2\alpha+m+1-l}\right\}$$
$$= (-1)^m\ m!\ x^{-\alpha-k}y^{2\alpha+m+1-l}P_m^{(\alpha,\alpha)}\left(1-\tfrac{2y}{x}\right). \qquad (4.6)$$

$$(T_k+\ T_l)^m\{x^{-m-\nu+\frac{1}{2}-k}\ y^{m+2\nu-l}\}$$
$$= \frac{(-1)^m\left(\nu+\frac{1}{2}\right)_m\ \ m!}{(2\nu)_m}x^{-\nu+\frac{1}{2}-k}y^{m+2\nu-l}C_m^{\nu}\left(-\tfrac{2y}{x}\right). \qquad (4.7)$$

$$(T_kT_l-\ T_n)^m\{x^{-m-k}y^{-m-l}z^{m+1-n}\}$$
$$= (m!)^2x^{-k}y^{-l}z^{m+1-n}Z_m\left(\frac{z}{xy}\right). \qquad (4.8)$$

$$\left(T_pT_q - T_kT_l\right)^m\{u^{-m-p}v^{-p-m+1-q}x^{m+1-k}y^{\xi-l}\}$$
$$= m!\ \frac{(p)_m x^{m+1-k}y^{\xi-l}}{u^p v^{p+q-1}} H_m\left(\xi, p, \frac{xy}{uv}\right). \tag{4.9}$$

$$(T_kT_l - T_n)^m\left\{x^{-m-\nu+\frac{1}{2}-k}y^{-l-b-m}z^{m+2\nu-n}\right\}$$
$$= \frac{\left(\nu+\frac{1}{2}\right)_m (1+b)_m z^{m+2\nu-n}}{x^{\nu+k-\frac{1}{2}}\ y^{l+b}} Z_m^{\nu}\left(b, \frac{z}{yx}\right) \tag{4.10}$$

$$(T_kT_l - T_n)^m\left\{x^{-m-a-k}\ y^{-l-b-m}z^{1+\alpha+\beta+m-n}\right\}$$
$$= \frac{(1+a)_m(1+b)_m z^{1+\alpha+\beta+m-n}}{x^{a+k}y^{b+l}} Z_m^{(\alpha,\beta)}\left(b, \frac{z}{xy}\right). \tag{4.11}$$

$$\left(T_pT_q - T_kT_l\right)^m\left\{u^{-m-p-\alpha}v^{-p-m+1-q}\ x^{m+1+\alpha+\beta-k}\ y^{\xi-l}\right\}$$
$$= m!\ \ (p)_m \frac{x^{1+\alpha+\beta+m-k}y^{\xi-l}}{u^{p+\alpha}\ \ v^{p+q-1}} H_m^{(\alpha,\beta)}\left(\xi, p, \frac{xy}{uv}\right). \tag{4.12}$$

$$(1+T_k)^m\{x^{m+1-k}\} = x^{m+1-k}\ y_m(2x) \tag{4.13}$$

$$\left(1+\frac{1}{T_k}\right)^m\{x^{a-1+m-k}\} = x^{a-1+m-k}y_m\left(a, b, \frac{-x}{b}\right). \tag{4.14}$$

$$(T_k+T_l)^m\left\{x^{1/2-m-k}y^{m-l}\right\} = (-1)^m\left(\frac{1}{2}\right)_m x^{1/2-k}y^{m-l}T_m\left(1-\frac{2y}{x}\right). \tag{4.15}$$

$$(T_k+T_l)^m\left\{x^{-\frac{1}{2}-m-k}y^{m+2-l}\right\} = (-1)^m\left(\frac{3}{2}\right)_m x^{-\frac{1}{2}-k}y^{m+2-l}U_m\left(1-\frac{2y}{x}\right). \tag{4.16}$$

$$\left(T_kT_lT_{l_1}T_{l_2}\cdots\ T_{l_q} - T_nT_{n_1}T_{n_2}\cdots\ T_{n_p}\right)^m$$
$$\left\{x^{-k-m}y^{\frac{1}{2}-l-m}v_1^{-b_1-l_1-m+1}v_2^{-b_2-l_2-m+1}\ldots\ v_q^{-b_{\{q\}}-l_q-m+1}\right.$$
$$\left. z^{m+1-n}w_1^{a_1-n_1}\ w_2^{a_2-n_2}\cdots w_p^{a_p-n_p}\right\}$$
$$= \left\{m!\ (-1)^{m\,q}\left(\frac{1}{2}\right)_m (b_1)_m(b_2)_m\ldots\left(b_q\right)_m x^{-k}y^{\frac{1}{2}-l}\right.$$
$$\left. v_1^{-b_1-l_1+1}v_1^{-b_1-l_1-m+1}\ \cdots\ v_q^{-b_q-l_q+1}w_1^{a_1-n_1}z^{m+1-n}\ w_2^{a_2-n_2}\cdots w_p^{a_p-n_p}\right\}$$
$$\times\ f_m\left[\begin{matrix}a_1,\ldots,a_p\ ; & \dfrac{z\,w_1w_2\cdots\ w_p}{x\,y\,v_1v_2\cdots\ v_q}\\ b_1,\ldots,b_q; & \end{matrix}\right] \tag{4.17}$$

where

$$f_m\left[\begin{matrix}a_1,\ldots,a_p\ ; & \dfrac{z\,w_1w_2\cdots\ w_p}{x\,y\,v_1v_2\cdots\ v_q}\\ b_1,\ldots,b_q; & \end{matrix}\right] = {}_{p+2}F_{q+2}\left[\begin{matrix}-n, n+1, a_1,\ldots,a_p\ ; & \\ 1, 1/2,\ b_1,\ldots,b_q; & x\end{matrix}\right].$$

$$\left(T_{l_1}T_{l_2}\cdots\ T_{l_k}, -\left(\frac{x}{k}\right)\right)^m\left\{y_1^{\frac{-\alpha-1}{k}-m-l_1+1}y_2^{\frac{-\alpha-2}{k}-m-l_2+1}\cdots\ y_k^{\frac{-\alpha-k}{k}-m-l_k+1}\right\}$$

$$= \frac{(\alpha+1)_{k\,m}\quad m!}{k^{km}(1+\alpha)_{2m}} Z_m^{\alpha}\left(\frac{x}{y_1y_2\cdots\ y_k};\ k\right). \tag{4.18}$$

$$(T_k + T_l)^m\{x^{\alpha-k}y^{\alpha-l}\} = m!\ x^{\alpha-k}y^{\beta-l}g_m^{(\alpha,\beta)}(x,y)\,. \qquad (4.19)$$

$$(T_k + T_l)^m\{x^{s+1-k}y^{1-l}\} = m!\ x^{s+1+m-k}y^{1-l}g_m^{(s)}\left(\frac{y}{x}\right). \qquad (4.20)$$

$$g(1 - T_l)^m\{y^{x-l}\} = m!\ x^{-m}y^{x-l}\phi_m\left(\frac{1}{y}\right). \qquad (4.21)$$

$$(T_k + T_l)^m\{x^{N-m+1-k}y^{-z-l}\}$$
$$= (-1)^m(-N)_m x^{N+1-k}y^{-z-l}K_m\left(z,\frac{x}{y},N\right). \qquad (4.22)$$

$$(T_k + T_l)^m\{x^{-\beta-m+1-k}y^{-z-l}\}$$
$$= (-1)^m x^{-\beta+1-k}y^{-z-l}M_m\left(z,\beta,\frac{x}{x-y}\right). \qquad (4.23)$$

$$\left(T_pT_l - \frac{y\,u}{z\,v}T_qT_n\right)^m\{u^{N-m+1-p}y^{\alpha-l}v^{-x-q}z^{\alpha+\beta+m+1-n}\}$$
$$= (-N)_m(\alpha+1)_m u^{N+1-p}y^{-\alpha-l}v^{-x-q}z^{\alpha+\beta+m+1-n}Q_m(x,\alpha,\beta,N). \qquad (4.24)$$

$$\left(T_l + \frac{y(1-e^{\lambda})}{z}T_n\right)^m\{y^{-m-1}z^{-x-n}\} = (-1)^m\, m!\ e^{m\lambda}y^{-l}z^{-x-n}l_m(x\,;\,\lambda). \qquad (4.25)$$

$$\left(1 - \frac{1}{ay}T_l\right)^m\{y^{-x-l}\} = \frac{1}{(-a)_m y^{x+l}}C_m^a(x). \qquad (4.26)$$

$$\left(T_k + \frac{2x}{y}T_l\right)^m\{x^{\gamma-m+1-k}y^{z-l}\} = (-1)^m\ m!\ x^{\gamma-k+1}y^{z-l}g_m(z,\gamma). \qquad (4.27)$$

$$(T_k + 1)^m\{x^{-a-2m+1-k}\} = (-1)^m\, m!\ x^{-a-m+1+k}R_m\left(a,\frac{1}{x}\right). \qquad (4.28)$$

$$\left(T_kT_lT_{l_n} + T_pT_qT_s\right)^m\left\{x^{-k+1}y^{1-\gamma-m-l}z^{-n+\beta}u^{-\frac{m}{2}-p}v^{-\frac{m}{2}+\frac{1}{2}-q}w^{\gamma-\beta-s}\right\}$$
$$= (m!)^2 2^{-m}(1-\nu-m)_m x^{-k+1+m}y^{\gamma-l+m}z^{\beta+m-n}$$
$$\times\ u^{-\frac{m}{2}-p}v^{-\frac{m}{2}+\frac{1}{2}-q}w^{\gamma-\beta-s}R_n\left(\beta,\nu,\sqrt{\frac{x\,y\,z}{uvw}}\right). \qquad (4.29)$$

$$\left(T_kT_lT_n + T_pT_qT_s\right)^m\left\{x^{-k+1}y^{-l+\alpha}z^{-n+\beta}u^{-\frac{m}{2}-p}v^{-\frac{m}{2}+\frac{1}{2}-q}w^{1-\alpha-\beta-m-s}\right\}$$
$$= (m!)^2 2^{-m}(\alpha+\beta)_m x^{-k+1+m}y^{-l+\alpha+m}z^{-n+\beta+m}$$
$$\times\ u^{\left\{-\frac{m}{2}-p\right\}}v^{\left\{-\frac{m}{2}+\frac{1}{2}-q\right\}}w^{\gamma-\beta-s}G_n\left(\alpha,\beta\,,\sqrt{\frac{x\,y\,z}{u\,v\,w}}\right) \qquad (4.30)$$

For justification regarding the formulae we discussed above, some proofs are elaborated here

Proof of (4.1)

$$(T_k + T_l)^m\left\{x^{-m-k}y^{m+1-l}\right\} = \sum_{r=0}^{m}\frac{(-m)_r\,(-1)^r}{r!}\ T_k^{m-r}(x^{-m-k})\ \ T_l^r\left(y^{m+1-l}\right)$$

$$= \sum_{r=0}^{m}\frac{(-m)_r\ (-1)^r}{r!}\ (-m)_{m-r}\,(x^{-k-r})\ \ (m+1)_r\,(y^{m+1-l+r})$$

$$= (-1)^m\, m!\ \ x^{-k}\,y^{m+1-l}\sum_{r=0}^{m}\frac{(-m)_r\ (m+1)_r}{r!\ (1)_r}\ \ \left(\frac{y}{x}\right)^r$$

$$= (-1)^m \ m! \ x^{-k} y^{m+1-l} \ {}_2F_1\left[\begin{matrix}-m & m+1; \\ & 1;\end{matrix} \ \frac{1-2y}{x}\right]$$

$$= (-1)^m m! \ x^{-k} y^{m+1-l} P_m\left(1 - \frac{2y}{x}\right).$$

Proof of (4.2)

$$(1 + T_k)^m\{x^{-(m+k)+r}\} = \sum_{r=0}^{m} \frac{(-m)_r (-1)^r}{r!} \ T_k^r \left(x^{-(m+k)+r}\right)$$

$$= \sum_{r=0}^{m} \frac{(-m)_r \ (-m+1)_r (-1)^r}{r!} \ x^{-(m+k)+2r}$$

$$= (x)^{-m-k} \ {}_2F_1\left[\begin{matrix}-\frac{m}{2} & -\frac{m}{2}+\frac{1}{2}; \\ - & ;\end{matrix} \ -4x^2\right]$$

$$= (x)^{-m-k} H_m\left(\frac{1}{2x}\right).$$

Proof of (4.7)

$$(T_k + T_l)^m\{x^{-m-\nu+\frac{1}{2}-k} \ y^{m+2\nu-l}\}$$

$$= \sum_{r=0}^{m} \frac{(-m)_r (-1)^r}{r!} \ T_k^{m-r}\left(x^{-m-\nu+\frac{1}{2}-k}\right) \ T_l^r(y^{m+2\nu-l})$$

$$= \left(-m-\nu+\frac{1}{2}\right)_m \ x^{-\nu+\frac{1}{2}-k} y^{m+2\nu-l}$$

$$\times \sum_{r=0}^{m} \frac{(-m)_r \ (m+2\nu)_r \ y^r}{r! \ \left(\nu+\frac{1}{2}\right)_r}$$

$$= \frac{(-1)^m \left(\nu+\frac{1}{2}\right)_m m!}{(2\nu)_m} \ x^{-\nu+\frac{1}{2}-k} \ y^{m+2\nu-l} \ C_m^{\nu}\left(1 - \frac{2y}{x}\right).$$

Proof of (4.8)

$$(T_kT_l - T_n)^m \{x^{-m-k} y^{-m-l} z^{m+1-n}\}$$

$$= \sum_{\{r=0\}}^{\{m\}} \frac{(-m)_r (-1)^r}{r!} \ T_k^{m-r}\left(x^{-m-\nu+\frac{1}{2}-k}\right)$$

$$\times T_l^{m-r}(y^{-m-l}) \ T_n^r(z^{m+1-n})$$

$$= \frac{(m!)^2 \ z^{m+1-n}}{x^k} \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r}{r! \ (1)_r (1)_r} \left(\frac{z}{xy}\right)$$

$$= (m!)^2 x^{-k} y^{-l} z^{m+1-n} Z_m^{\nu}\left(\frac{Z}{xy}\right).$$

Proof of (4.9)

$$\left(T_p \ T_q - T_k T_l\right)^m \left\{u^{-m-p} v^{-p-m+1-q} x^{m+1-k} y^{\xi-l}\right\}$$

$$= \sum_{r=0}^{m} \frac{(-m)_r(-1)^r}{r!} \left\{T_p^{m-r}\left(u^{-m-p}\right) T_q^{m-r}\left(v^{-p-m+1-q}\right) T_k^r \left(x^{m+1-k}\right) \ T_l^r\left(y^{\xi-l}\right)\right\}$$

$$= (-m)_m (-p - m + 1)_m \ \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r (\xi)_r}{r! \ (1)_r \ (p)_r} \times \left(\frac{xy}{uv}\right) \frac{x^{m+1-k} y^{\xi-l}}{u^p \ v^{p+q-1}}$$

$$= \frac{m! \ (p)_m x^{m+1-k} y^{\xi-l}}{u^p \ v^{p+q-1}} H_m\left(\xi, p, \frac{xy}{uv}\right).$$

Proof of (4.19)

$$(T_k + T_l)^m\{x^{\alpha-k} y^{\beta-l}\} = \sum_{r=0}^{m} \frac{(-m)_r (-1)^r}{r!} \ \{T_k^{m-r}(x^{\alpha-k}) \ T_l^r\left(y^{\beta-l}\right)\}$$

$$= (\alpha)_m x^{\alpha-k+m} y^{\beta-l} \sum_{r=0}^{m} \frac{(-m)_r (-1)^r}{r! \ (1-\alpha-m)_r} \left(\frac{y}{x}\right)^r$$

$$= m! \ x^{\alpha-k} \ y^{\beta-l} g_m^{(\alpha,\beta)}(x, y).$$

Proof of (4.20)

$$(T_k + T_l)^m \left\{x^{s+1-k}\, y^{1-l}\right\} = \sum_{r=0}^{m} \frac{(-m)_r\,(-1)^r}{r!} \left\{T_k^{m-r}(x^{s+1-k})\ \ T_l^r\left(y^{1-l}\right)\right\}$$

$$= (s+1)_m\ x^{s+1-k+m}\, y^{1-l} \sum_{r=0}^{m} \frac{(-m)_r}{r!\,(-s-m)_r}\left(\frac{y}{x}\right)^r$$

$$= m!\ x^{s+1+m-k}\, y^{1-l}\, g_m^{(s)}\left(\frac{y}{x}\right).$$

Proof of (4.21)

$$(1 - T_l)^m \{\, y^{x-l}\}$$

$$= \sum_{r=0}^{m} \frac{(-m)_r}{r!} \left\{T_l^r\left(y^{x-l}\right)\right\} \quad (\,using\ \ 2.3)$$

$$= \sum_{r=0}^{m} \frac{(-m)_r}{r!}\{(x)_r\ (y^{x-l+r})\}$$

$$= y^{x-l}\ {}_2F_0\left[\begin{matrix}-m, & x; & \\ - & ; & \end{matrix} y\right]$$

$$= m!\ \ x^{-m}\ y^{x-l}\ \phi_m\left(\frac{1}{y}\right).$$

Proof of (4.22)

$$(T_k + T_l)^m\{x^{N-m+1-k}\ y^{-z-l}\} = \sum_{r=0}^{m} \frac{(-m)_r\,(-1)^r}{r!}\left\{T_k^{m-r}\left(x^{N-m+1-k}\right)\ T_l^r\left(y^{-z-l}\right)\right\}$$

$$= (s+1)_m\, x^{s+1-k+m}\ \ y^{1-l} \sum_{r=0}^{m} \frac{(-m)_r\,(-1)^r}{r!\,(-s-m)_r}\left(\frac{y}{x}\right)^r$$

$$= m!\ \ x^{s+1+m-k}\, y^{1-l}\, g_m^{(s)}\left(\frac{y}{x}\right).$$

Proof of (4.26)

$$\left(1 - \frac{1}{ay}\ T_l\right)^m \{y^{-x-l}\} = \sum_{r=0}^{m} \frac{(-m)_r}{r!}\left(\frac{1}{ay}\right)^r\ T_l^r(y^{-x-l})$$

$$= \frac{1}{(-a)^m}\ y^{x-l}\ \ {}_2F_0\left[\begin{matrix}-m, & -x; & \\ - & ; & \end{matrix} \frac{1}{a}\right]$$

$$= \frac{1}{(-a)_m y^{x+l}}\ C_m^a(x).$$

Proof of (4.27)

$$\left(T_k + \frac{2x}{y}\ T_l\right)^m \left\{x^{\gamma-m+1-k}\ y^{z-l}\right\}$$

$$= T_k^{m-r}(x^{\gamma-m+1-k})\ \ \left(\frac{2x}{y}\right)^r\ \ T_l^r(y^{z-l})$$

$$= \sum_{r=0}^{m} \frac{(-m)_{\{r\}}\ (-\gamma)_m}{r!\,(-\gamma)_r}\ \ 2^r\ (z)_r\ x^{\gamma+1-k}\ y^{z-l}$$

$$= (-1)^m(-\gamma)_m\ \ x^{\gamma-k+1}\ y^{z-l}\ \ {}_2F_1\left[\begin{matrix}-m, & z; & \\ \gamma & ; & \end{matrix} 2\right]$$

$$= (-1)^m\ \ (-\gamma)_m\ x^{\{\gamma-k+1\}}\ y^{z-l}\ g_m(z,\gamma).$$

Proof of (4.28)

$$(T_k + 1)^m \left\{x^{\{-a-2m+1-k\}}\right\} = \sum_{r=0}^{m} \frac{(-m)_r\,(-1)^r}{r!}\ \{T_k^{m-r}(x^{-a-2m+1-k})\}$$

$$= (-1)^m\ m!\ x^{-a-m+1+k}\ \ R_m(a, 1/x).$$

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