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Research Article

Symmetric functions for *k*-Pell Numbers at negative indices

Ali Boussayoud*

LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria

* Corresponding author, e-mail: aboussayoud@yahoo.fr

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Abstract: In this paper, we introduce a new operator in order to derive some properties of homogeneous symmetric functions. By making use of the proposed operator, we give some new generating functions for k -Fibonacci and k -Pell numbers at negative indices and product of numbers at negative indices and Chebychev polynomials of first and second kind.

Keywords: Symmetric functions; Generating functions; k -Pell numbers at negative indices.

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1. INTRODUCTION AND PRELIMINARIES

The well-known Fibonacci (and Lucas) sequence is one of the sequences of positive integers that have been studied over several years. Many authors are dedicated to study this sequence, such as the work of Hoggatt, in [11] and Vorobiov, in [15], among others. Fibonacci numbers F_n are defined by the recurrence relation

$$\begin{cases} F_0 = 1, F_1 = 1 \\ F_{n+1} = F_n + F_{n-1},\ n \geq 1 \end{cases}.$$

Pell numbers P_n are defined by the recurrence relation

$$\begin{cases} P_0=0, P_1=1 \\ P_{n+1}=2P_n+P_{n-1},\ n\geq 1 \end{cases}.$$

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the k -Fibonacci Numbers. For any positive real number k , the k -Fibonacci sequence, say $(F_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by [13]

$$\begin{cases} F_{k,0}=1, F_{k,1}=1 \\ F_{k,n+1}=kF_{k,n}+F_{k,n-1},\ n\geq 1 \end{cases}.$$

In [12], k -Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [12, 13].

For any positive real number k , the k -Pell Numbers, say $(P_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by [8]

$$\begin{cases} P_{k,0}=0, P_{k,1}=1 \\ P_{k,n+1}=2P_{k,n}+kP_{k,n-1},\ n\geq 1 \end{cases}.$$

In this contribution, we shall define a new useful operator denoted by $\delta^{k+1}_{p_1p_2}$ for which we can formulate, extend and prove new results based on our previous ones [1, 2, 6]. In order to determine generating functions of the product of k -Fibonacci and k -Pell numbers at negative indices and Chebychev polynomials of first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results

Definition 1.1. *[6] Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form*

$$\frac{\Pi_{p\varepsilon P}(1-pt)}{\Pi_{b\varepsilon B}(1-bt)}=\sum_{n=0}^{\infty}S_n(B-P)t^n, \tag{1.2}$$

with the condition $S_n(B-P)=0$ for $n<0$.

Equation (1.2) can be rewritten in the following form

$$\sum_{n=0}^{\infty}S_n(B-P)t^n=\left(\sum_{n=0}^{\infty}S_n(B)t^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-P)t^n\right),$$

where

$$S_n(B-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(B).$$

Definition 1.2. *[4] Given a function* f *on* $\mathbb{R}^n$ *, the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1,\cdots,p_i,p_{i+1},\cdots p_n) - f(p_1,\cdots p_{i-1},p_{i+1},p_i,p_{i+2}\cdots p_n)}{p_i - p_{i+1}}.$$

Definition 1.3. *The symmetrizing operator* $\delta^k_{e_1 e_2}$ *is defined by*

$$\delta^k_{p_1p_2}(g(p_1)) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathrm{N}.$$

Proposition 1.1. *[5] Let* $P = \{p_1, p_2\}$ *an alphabet, we define the operator* $\delta^k_{p_1p_2}$ *as follows*

$$\delta^k_{p_1p_2} g(p_1) = S_{k-1}(p_1+p_2)g(p_1) + p_2^k \partial_{p_1p_2} g(p_1), \text{ for all } k \in \mathrm{N}.$$

Proposition 1.2. *[1] The relations*

$$1) F_{k,-n} = (-1)^{n+1} F_{k,n},$$
$$2) P_{k,-n} = (-1)^{n+1} P_{k,n}.$$

hold for all $n \geq 0$.

2. THEOREM AND PROOF

In our main result, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 2.1. *Given two alphabets* $P = \{p_1, p_2\}$ *and* $B = \{b_1, b_2, ..., b_n\}$, *we have*

$$\sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+k+1})t^n = \frac{\sum\limits_{n=0}^{\infty} S_n(-B)\delta^{k+1}_{p_1p_2}(p_2^n)t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}. \quad (2.1)$$

Proof. Let $\sum_{n=0}^{\infty} S_n(B)t^n$ and $\sum_{n=0}^{\infty} S_n(-B)t^n$ be two sequences such that $\sum\limits_{n=0}^{\infty} S_n(B)t^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-B)t^n}$. On one hand, since $g(p_1) = \sum\limits_{n=0}^{\infty} S_n(B)p_1^n t^n$, we have

$$\delta_{p_1p_2}^{k+1} g(p_1) = \delta_{p_1p_2}^{k+1}\left(\sum_{n=0}^{\infty} S_n(B)p_1^n t^n\right)$$

$$= \frac{p_1^{k+1}\sum_{n=0}^{\infty} S_n(B)p_1^n t^n - p_2^{k+1}\sum_{n=0}^{\infty} S_n(B)p_2^n t^n}{p_1 - p_2}$$

$$= \sum_{n=0}^{\infty} S_n(B)\left(\frac{p_1^{n+k+1} - p_2^{n+k+1}}{p_1 - p_2}\right)t^n$$

$$= \sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+k+1})t^n.$$

which is the right-hand side of (2.1). On the other part, since

$$g(p_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n},$$

we have

$$\delta_{p_1p_2}^{k+1} g(p_1) = \frac{p_1^{k+1}\prod_{b\in B}(1-bp_2)t - p_2^{k+1}\prod_{b\in B}(1-bp_1 t)}{(p_1-p_2)\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}.$$

Using the fact that: $\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n = \prod_{b\in B}(1-bp_1 t)$, then

$$\delta_{p_1p_2}^{k+1} g(p_1) = \frac{\sum_{n=0}^{\infty} S_n(-B)\frac{p_1^{k+1}p_2^n - p_2^{k+1}p_1^n}{p_1-p_2}t^n}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}$$

$$= \frac{\sum_{n=0}^{\infty} S_n(-B)\delta_{p_1p_2}^{k+1}(p_2^n)t^n}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}.$$

This completes the proof.

3. ON THE SYMMETRIC FUNCTIONS

We now derive new generating functions of the products of some well-known numbers and polynomials. Indeed, we consider Theorem 2.1 in order to derive k -Fibonacci and k -Pell numbers at negative indices and Tchebychev polynomials of first and second kind and the symmetric functions for $k = 0$.

Theorem 3.1. *[4] Given two alphabets* $P = \{p_1, p_2\}$ *and* $B = \{b_1, b_2, b_3\}$, *we have*

$$\sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+1})t^n = \frac{S_0(-B)-p_1p_2S_2(-B)t^2-p_1p_2S_3(-B)\left(p_1+p_2\right)t^3}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}. \tag{3.1}$$

Case 1: *Replacing* p_2 *by* $(-p_2)$ *and assuming that* $p_1p_2=1,\ p_1-p_2=k$ *in Theorem 3.1, we have the following theorem*

Theorem 3.2. *We have the following a new generating function of both* k *-Fibonacci numbers at negative indices and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_{k,-n}t^n = \frac{kS_3(-B)t^3+S_2(-B)t^2-1}{\prod_{i=1}^{3}\left(1+kb_it-b_i^2t^2\right)}. \tag{3.2}$$

- Put $k=1$ in the relationship (3.2) we have

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_{-n}t^n = \frac{b_1b_2b_3t^3+(b_1b_2+b_1b_3+b_2b_3)t^2-1}{\prod_{i=1}^{3}(1+b_it-b_i^2t^2)},$$

which representing a new generating function of Fibonacci numbers at negative indices and symmetric functions in several variable ⋅

Setting $b_3=0$ and replacing b_2 by $(-b_2)$ in (3.2), and assuming $b_1-b_2=k$; $b_1b_2=1$, we deduce the following theorems.

Theorem 3.3. *For* $n\in\mathbb{N}$ *, the new generating function of the product of* k *-Fibonacci numbers and* k *-Fibonacci numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} F_{k,n}F_{k,-n}t^n = \frac{t^2-1}{1+k^2t-2(k^2+1)t^2+k^2t^3+t^4}. \tag{3.3}$$

- Put $k=1$ in the relationship (3.3) we have

$$\sum_{n=0}^{\infty} F_nF_{-n}t^n = \frac{t^2-1}{1+t-4t^2+t^3+t^4},$$

which representing a new generating function of the product of Fibonacci numbers and Fibonacci numbers at negative indices.

Theorem 3.4. *For* $n\in\mathbb{N}$ *, the new generating function of the product of* k *-Fibonacci numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} F_{k,-n}^2t^n = \frac{1-t^2}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}.$$

Case 2: *Replacing* p_2 *by* $(-p_2)$ *and assuming that* $p_1p_2=k$, $p_1-p_2=2$ *in Theorem 3.1, we have the following.*

Theorem 3.5. *We have the following a new generating function of both* k *-Pell numbers at negative indices and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_{n-1}(b_1+b_2+b_3)P_{k,-n}t^n=\frac{t-kS_2(-B)t^3+2kS_3(-B)t^4}{\prod_{i=1}^{3}\left(1+2b_it-kb_i^2t^2\right)}. \quad (3.4)$$

- Put $k=1$ in the relationship (3.4) we have

$$\sum_{n=0}^{\infty} S_{n-1}(b_1+b_2+b_3)P_{-n}t^n=\frac{t-S_2(-B)t^3+2S_3(-B)t^4}{\prod_{i=1}^{3}\left(1+2b_it-b_i^2t^2\right)},$$

which representing a new generating function of Pell numbers at negative indices and symmetric functions in several variables.

Setting $b_3=0$ and replacing b_2 by $(-b_2)$ in (3.4), and assuming $b_1-b_2=2$; $b_1b_2=k$, we deduce the following theorem.

Theorem 3.6. *For* $n\in \mathrm{N}$ *, the new generating function of the product of* k *-Pell numbers and* k *-Fibonacci numbers at negative indices is given by*

$$\sum_{n=0}^{\infty} P_{k,n}P_{k,-n}t^n=\frac{2t-2kt^2}{1+4t-(2k^2+8k)t^2+4k^2t^3+k^4t^4}. \quad (3.5)$$

- Put $k=1$ in the relationship (3.5) we have

$$\sum_{n=0}^{\infty} P_nP_{-n}t^n=\frac{2t-2kt^2}{1+4t-10t^2+4t^3+t^4},$$

which representing a new generating function of the product of Pell numbers and Pell numbers at negative indices.

Theorem 3.7. *For* $n\in \mathrm{N}$ *, the new generating function of the product of* k *-Pell numbers is given by*

$$\sum_{n=0}^{\infty} P_{k,-n}^2t^n=\frac{2t+2kt^2}{1-4t-(2k^2+8k)t^2-4k^2t^3+k^4t^4}.$$

Case 3: *Replacing* p_2 *by* $-p_1$, *and assuming that* $p_1p_2=1$, $p_1-p_2=x$ *in Theorem 3.1, we have the following a new generating function Fibonacci polynomials of second kind and the symmetric functions in several variables, as follows*

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)F_n(x)t^n=\frac{1+S_2(-B)t^2+xS_3(-B)t^3}{\prod_{i=1}^{3}\left(1-kb_it-b_i^2t^2\right)}. \quad (3.6)$$

Setting $b_3 = 0$ and replacing b_2 by $(-b_2)$ in (3.6), and assuming $b_1 - b_2 = k$; $b_1 b_2 = 1$ and $b_1 - b_2 = 2$; $b_1 b_2 = k$ respectively, we deduce the following theorems.

Theorem 3.8. *The generating function of the product of Fibonacci polynomials and k-Fibonacci numbers as*

$$\sum_{n=0}^{\infty} F_{k,n} F(x) = \frac{1-t^2}{1-kxt-(k^2+x^2+2)t^2-kxt^3+t^4}.$$

Theorem 3.9. *The new generating function of the product of Fibonacci polynomials and k-Pell numbers as*

$$\sum_{n=0}^{\infty} P_{k,n} F(x) = \frac{t-kt^3}{1-2xt-(4+kx^2+2k)t^2-kxt^3+k^2t^4}.$$

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4. CONCLUSIONS

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric fonctions.The obtained results agree with the results obtained in some previous works.

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